Remark. S. Kôno also obtains a similar result in the case of complex $C_{p^m q^n}$-representations.

For details of the proof, see [34] and [35]. The idea is to decompose (V, W) into primitive pairs (V_i, W_i).

Definition. A pair of representations (V, W) is called *primitive* if V and W cannot be decomposed into $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$ such that (V_i, W_i), $V_i \neq 0$, $W_i \neq 0$, satisfies (C_{V_i, W_i}), $i = 1, 2$.

Then, by constructing a G-isovariant map $f_i : V_i \to W_i$, we have a G-isovariant map $f = \oplus_i f_i : V \to W$.

Example 4.5. The following are examples of primitive pairs of C_n-representations, and there exist isovariant maps between the representations. Suppose that p, q, r are pairwise coprime integers greater than 1.

(1) (U_k, U_l) when $(k, n) = (l, n) = 1$.

(2) $(U_1, U_p \oplus U_q)$ when pq divides n.

(3) $(U_p \oplus U_q,\ U_{p^2} \oplus U_{pq})$ when $p^2 q$ divides n.

(4) $(U_p \oplus U_q \oplus U_r,\ U_1 \oplus U_{pq} \oplus U_{qr} \oplus U_{pr})$ when pqr divides n.

In the cases (1)-(3), one can define a C_n-isovariant map concretely; however, in case (4), equivariant obstruction theory is used. We illustrate it in Section 5.

On the other hand, there exists a group not having the complete IB-property.

Theorem 4.6 ([35]). *Let D_n be the dihedral group of order $2n$ $(n \geq 3)$. Every D_n $(n \neq 3, 4, 6)$ does not have the complete IB-property.*

The dihedral group D_n has the following presentation:

$$D_n = \langle a, b \,|\, a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle.$$

One has the normal cyclic subgroup $C_m = \langle a^{n/m} \rangle$ of D_n for every divisor m of n, and there are n/m dihedral subgroups $\langle a^{n/m}, b \rangle, \langle a^{n/m}, b^2 \rangle, \ldots, \langle a^{n/m}, a^{n/m-1} b \rangle$ containing C_m as a subgroup of index 2. If n/m is odd, then these are all conjugate in D_n. As a representative of their conjugacy class, we take $D_m = \langle a^{n/m}, b\rangle$. If n/m is even, then there are two conjugacy classes. As representatives, we take $D_m = \langle a^{n/m}, b\rangle$ and $D'_m = \langle a^{n/m}, ab\rangle$.

Let $T_k = \mathbb{C}$, $1 \le k < n/2$, be the D_n-representation on which D_n acts by $a \cdot z = \xi^k z$, $b \cdot z = \overline{z}$, $z \in S_k$, where $\xi = \exp(2\pi\sqrt{-1}/n)$. These T_k are all (nonisomorphic) 2-dimensional irreducible representations over $\mathbb{R}$ [45]. It follows that Ker $T_k = C_{(k,n)}$. and

$$\mathrm{Iso}\ T_k = \{D_n, \langle a^{n/(k,n)}, a^t b\rangle, \langle a^{n/(k,n)}\rangle \mid 0 \le t \le n-1\}.$$

Note also that

$$\dim T_k^H = \begin{cases} 2 \text{ if } H \le C_{(k,n)} \\ 1 \text{ if } H \text{ is conjugate to } D_{(k,n)} \text{ or } D'_{(k,n)} \\ 0 \text{ otherwise.} \end{cases}$$

Proof of Theorem 4.6. Let k be an integer prime to n with $1 < k < n/2$. Consider a pair $(T_1,\ T_k)$ of representations of D_n. It is easily seen that $(T_1,\ T_k)$ satisfies conditions (C_{T_1,T_k}) and (I_{T_1,T_k}). We show that there is no D_n-isovariant map from T_1 to T_k. Suppose that there is a D_n-isovariant map from T_1 to T_k for some k; then, by normalization, one has a D_n-isovariant map $f : ST_1 \to ST_k$. Note that $ST_1^{>1} = ST_k^{>1} = \{\exp(\pi t\sqrt{-1}/n) \mid 0 \le t \le n-1\}$. Take $x = 1$ and $y = \exp(\pi\sqrt{-1}/n)$, then the isotropy subgroup at x in ST_1 is $\langle b\rangle$, and also the isotropy subgroup at y in ST_1 is $\langle ab\rangle$. Since $ST_k^{\langle b\rangle} = \{\pm 1\} \subset \mathbb{C}$, it follows that $f(1) = \pm 1$. Composing, if necessary, the antipodal map $z \mapsto -z$ on ST_k with f, we may assume $f(1) = 1$. Let A be the shorter arc joining x with y in ST_1. Since every point of the interior of A has trivial isotropy subgroup, it follows that $f(A \setminus \{x, y\})$ is contained in $ST_k \setminus ST_k^{>1}$; hence $f(y)$ must be y or $\overline{y}$. However the isotropy subgroup at y (resp. $\overline{y}$) in ST_k is equal to $\langle a^r b\rangle$ (resp. $\langle a^{-r} b\rangle$), where r is a positive integer with $kr \equiv 1 \bmod n$, but it is not equal to $\langle ab\rangle$, since $k \not\equiv \pm 1 \bmod n$. This contradicts the isovariance of f. Thus the proof is complete. □

5 The existence of isovariant maps from a rational homology sphere with pseudofree S^1-action to a linear S^1-sphere

Let $G = S^1$ ($\subset \mathbb{C}$). Let T_i ($= \mathbb{C}$) be the irreducible representation of S^1 defined by $g \cdot z = g^k z$. Let M be a rational homology sphere with *pseudofree* S^1-action.

Definition (Montgomery-Yang [28]). An S^1-action on M is *pseudofree* if

(1) the action is effective, and

(2) the singular set $M^{>1} := \bigcup_{1 \neq H \leq S^1} M^H$ is not empty and consists of finitely many exceptional orbits.

Here an orbit $G(x)$ is called exceptional if $G(x) \cong S^1/D$, $(1 \neq D < S^1)$ [6].

Remark. Other meanings for the term "pseudofree action" appear in the literature.

Example 5.1. Let $V = T_p \oplus T_q \oplus T_r$. Then the S^1-action on SV is pseudofree. Indeed it is clearly effective, and

$$\begin{aligned} SV^{>1} =& ST_p \coprod ST_q \coprod ST_r \\ \cong& S^1/C_p \coprod S^1/C_q \coprod S^1/C_r. \end{aligned}$$

Remark. There are many "exotic" pseudofree S^1-actions on high-dimensional homotopy spheres [28], [42].

Then the following isovariant Borsuk-Ulam type result can be verified.

Theorem 5.2 ([33]). *Let M be a rational homology sphere with* pseudofree *S^1-action and SW a linear S^1-sphere. There is an S^1-isovariant map $f : M \to SW$ if and only if*

(I): $\operatorname{Iso} M \subset \operatorname{Iso} SW$,

(PF1): $\dim M - 1 \leq \dim SW - \dim SW^H$ *when H is a nontrivial subgroup which is contained in some $D \in \operatorname{Iso} M$,* (PF2): $\dim M + 1 \leq \dim SW - \dim SW^H$ *when* H *is a nontrivial subgroup which is not contained in any* $D \in \operatorname{Iso} M$.

We give some examples. Let p, q, r be pairwise coprime integers greater than 1.

Example 5.3. There is no S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Proof. Condition (PF1) is not fulfilled. □

Remark. There is an S^1-equivariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Example 5.4. There is an S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Proof. One can see that $\operatorname{Iso} M = \{1, C_p, C_q, C_r\}$ and

$$\operatorname{Iso} SW = \{1, C_p, C_q, C_r, C_{pq}, C_{qr}, C_{rp}\}.$$

Hence it is easily seen that (PF1) and (PF2) are fulfilled and $\operatorname{Iso} M \subset \operatorname{Iso} SW$. By the theorem above, there is an S^1-isovariant map. □

From this, we obtain an isovariant map in the case of Example 4.5(4).

Corollary 5.5. *There is an* C_{pqr}*-isovariant map*

$$f : S(U_p \oplus U_q \oplus U_r) \to S(U_1 \oplus U_{pq} \oplus U_{qr} \oplus U_{rp}).$$

Proof. By restricting f in Example 5.4 to the C_{pqr}-action, one has the desired map. □

5.1 Proof of Theorem 5.2 (outline)

We shall give an outline of the proof of Theorem 5.2. Full details can be found in [33]. Set $SW_{\text{free}} := SW \setminus SW^{>1}$. Note that S^1 acts freely on SW_{free}. Let N_i be an S^1-tubular neighborhood of each exceptional orbit in M. By the slice theorem, N_i is identified with $S^1 \times_{D_i} DU_i$ $(1 \le i \le r)$, where D_i is the isotropy group of the exceptional orbit and U_i is the slice D_i-representation. Set $X := M \setminus (\coprod_i \operatorname{int} N_i)$. Note that S^1 acts freely on X.

The "only if" part is proved by the (isovariant) Borsuk-Ulam theorem. Indeed for (PF1), take a point $x \in M$ with $G_x = D$ and a D-invariant closed neighborhood B of x which is D-diffeomorphic to some unit disk DV. Hence we obtain an H-isovariant map $f_{|SV} : SV \to SW$ by restriction. Applying the isovariant Borsuk-Ulam theorem to f, we obtain (PF1).

We next show (PF2). Since f is isovariant, one sees that f maps M into $SW \setminus SW^H$. Since $SW \setminus SW^H$ is S^1-homotopy equivalent to $S(W^{H^\perp})$, one obtains an S^1-map $g : M \to S(W^{H^\perp})$. By Corollary 2.3, condition (PF2) follows.

To show the converse, we use the equivariant obstruction theory. We recall the following result.

Lemma 5.6. *There is an S^1-isovariant map $\tilde{f}_i : N_i \to SW$.*

Proof. Let $N_i = N \cong_G S^1 \times_D DV \subset M$, where D is the isotropy group of the exceptional orbit and V is the slice representation. Similarly take a closed S^1-tubular neighborhood N' of an exceptional orbit with isotropy group D, and set $N' \cong_G S^1 \times_D DV' \subset SW$. By (PF1), one sees that $\dim SV + 1 \le \dim SV' - \dim SV'^{>1}$. Since D acts freely on SV, there is a D-map $g : SV \to SV' \setminus SV^{>1} \subset SW$ by Corollary 2.8, which leads to a D-isovariant map $g : SV \to SW$. Taking a cone, we have a D-isovariant map $\tilde{g} : DV \to DV'$, and hence an S^1-isovariant map $\overline{f} = S^1 \times_D \tilde{g} : N \to N' \subset SW$. □

Set $f_i := \tilde{f}_i|_{\partial N_i} : \partial N_i \to SW_{\text{free}}$, and $f := \coprod_i f_i : \partial X \to SW_{\text{free}}$. If f is extended to an S^1-map $F : X \to SW_{\text{free}}$, by gluing the maps, we obtain an S^1-isovariant map

$$F \cup (\coprod_i \tilde{f}_i) : M \to SW.$$

Thus we need to investigate the extendability of an S^1-map $f : \partial X \to SW_{\text{free}}$ to $F : X \to SW_{\text{free}}$. Equivariant obstruction theory [10] answers this question. A standard computation shows

Lemma 5.7 ([33], [38]). *Set $d = \dim SW - \dim SW^{>1}$.*

(1) *SW_{free} is $(d-2)$-connected and $(d-1)$-simple.*

(2) *$\pi_{d-1}(SW_{\text{free}}) \cong H_{d-1}(SW_{\text{free}})) \cong \oplus_{H\in\mathcal{A}}\mathbb{Z}$, where*

$$\mathcal{A} := \{H \in \operatorname{Iso} SW \,|\, \dim SW^H = \dim SW^{>1}\}$$

and the generators are represented by $S(W^{H^\perp})$, $H \in \mathcal{A}$.

By noticing that $\dim M - 1 \le d$ by (PF1) and (PF2), the obstruction $\mathfrak{o}_{S^1}(f)$ to the existence of an S^1-map $F : X \to SW_{\text{free}}$ lies in the equivariant cohomology group

$$\mathfrak{H}^d_{S^1}(X, \partial X; \pi_{d-1}(SW_{\text{free}})) \cong H^d(X/S^1, \partial X/S^1; \pi_{d-1}(SW_{\text{free}})).$$

If $\dim M - 1 < d$ (i.e., $\dim X/S^1 < d$), then one sees that

$$H^*(X/S^1, \partial X/S^1; \pi_{*-1}(SW_{\text{free}})) = 0$$

by dimensional reasons. Hence the obstruction vanishes and there exists an extension $F : X \to SW_{\text{free}}$.

We hereafter assume that $\dim M - 1 = d$ (i.e., $\dim X/S^1 = d$). The computation of the obstruction is executed by the multidegree.

Definition. Let $N = S^1 \times_D DU \subset M$, $1 \neq D \in \operatorname{Iso} M$. Assume that $\dim M - 1 = \dim U = d$. Let $f : \partial N \to SW_{\text{free}}$ be an S^1-map, and consider the D-map $\overline{f} = f|_{SU} : SU \to SW_{\text{free}}$. Then the multidegree of f is defined by

$$\operatorname{mDeg} f := \overline{f}_*([SU]) \in \oplus_{H\in\mathcal{A}}\mathbb{Z},$$

under the natural identification $H_{d-1}(SW_{\text{free}})$ with $\oplus_{H\in\mathcal{A}}\mathbb{Z}$.

The obstruction $\mathfrak{o}_{S^1}(f)$ is described by the multidegree as follows.

Proposition 5.8 ([33]). *Let $F_0 : X \to SW_{\text{free}}$ be a fixed S^1-map; this map always exists, however, it is not necessary to extend it to an isovariant map on M. Set $f_{0,i} = F_0|_{\partial N_i}$. Then*

$$\mathfrak{o}_{S^1}(f) = \sum_{i=1}^{r} (\text{mDeg}\, f_i - \text{mDeg}\, f_{0,i})/|D_i|,$$

under the natural identification $H_{d-1}(SW_{\text{free}})$ with $\oplus_{H \in \mathcal{A}} \mathbb{Z}$.

Remark. It follows from the equivariant Hopf type result [33] that

$$\text{mDeg}\, f_i - \text{mDeg}\, f_{0,i} \in \oplus_{H \in \mathcal{A}} |D_i| \mathbb{Z}.$$

In addition, the following extendability result is known.

Proposition 5.9 ([33]). *Let $N = S^1 \times_D DV$ be as before and $f : \partial N \to SW_{\text{free}}$ be an S^1-map. Set* $\text{mDeg}\, f = (d_H(f))$.

(1) *$f : \partial N \to SW_{\text{free}}$ is extendable to an S^1-isovariant map $\tilde{f} : N \to SW$ if and only if $d_H(f) = 0$ for any $H \in \mathcal{A}$ with $H \not\leq D$.*

(2) *For any extendable f and for any $(a_H) \in \oplus_{H \in \mathcal{A}} |D| \mathbb{Z}$ satisfying $a_H = 0$ for $H \in \mathcal{A}$ with $H \not\leq D$, there exists an S^1-map $f' : \partial N \to SW_{\text{free}}$ such that f' is extendable to an S^1-isovariant map $\tilde{f}' : N \to SW$ and* $\text{mDeg}\, f' = \text{mDeg}\, f + (a_H)$.

Using these propositions, one can see that there are S^1-isovariant maps $f_i : \partial N_i \to SW$ such that $\coprod_i f_i$ extends both on X and on $\coprod_i N_i$ as isovariant maps. Thus an isovariant map from M to SW is constructed.

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Received 15 March 2010 and in revised form 12 September 2010.

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Far East Journal of Mathematical Sciences (FJMS)
Volume 38, Number 2, 2010, Pages 205-216
This paper is available online at http://www.pphmj.com

THE SMITH HOMOLOGY AND BORSUK-ULAM TYPE THEOREMS

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2010 Mathematics Subject Classification: 57S10, 57S17, 55M20, 55M35, 03C64.

Keywords and phrases: Borsuk-Ulam theorem, finite groups, continuous C_k -maps, o-minimal, definable C_k -maps, real closed fields.

The fourth author is partially supported by Kyoto Sangyo University Research Grants.

Received January 27, 2010

Abstract

Let k be a positive integer greater than 1 and C_k be the cyclic group of order k. Let X be an arcwise connected free C_k-space and Y be a Hausdorff free C_k-space. If there exists a positive integer n such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \le q \le n$ and $H_{n+1}(Y/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then there is no continuous C_k-map from X to Y.

We also prove a definable version of this topological version in an o-minimal expansion of $\mathcal{N} = (R, +, \cdot, <, ...)$ of a real closed field R.

1. Introduction

Let k be a positive integer greater than 1 and C_k be the cyclic group of order k. Let $\mathbb{S}^n$ be the n-dimensional unit sphere of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with the antipodal C_2-action. From the viewpoint of transformation groups, the classical Borsuk-Ulam theorem states that if there exists a continuous C_2-map from $\mathbb{S}^n$ to $\mathbb{S}^m$, then $n \le m$. There are several equivalent statements of it and many related generalizations (e.g., [2], [13], [14], [15], [17]).

The classical Borsuk-Ulam theorem is generalized to topological spaces by several authors. For example, Walker [21], Pergher et al. [18]. They prove non-existence of continuous C_2-maps between free C_2-spaces under certain homological conditions on the free C_2-spaces. Essentially they use the Smith-Gysin exact sequence in their proof. If k is a positive integer greater than 1, then several C_k-versions of the classical Borsuk-Ulam theorem are discussed in Kobayashi [11] and Hemmi et al. [7].

In this paper, we use the Smith homology (c.f. [10]) which is a useful simple tool to study C_k-versions of the classical Borsuk-Ulam theorem in the topological setting and the definable setting. The Smith exact sequence which is expressed by using the Smith homology is a generalization of the Smith-Gysin exact sequence. By using this, we can give a simple proof of a C_k-version of the classical Borsuk-Ulam theorem. In this paper, we prove the following generalized Borsuk-Ulam theorem which is a generalization of [21], [18], [11] and [7].

Theorem 1.1. *Let X be an arcwise connected free C_k-space and Y be a Hausdorff free C_k-space. If there exists a positive integer n such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \le q \le n$ and $H_{n+1}(Y/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then there is no continuous C_k-map from X to Y. Here this homology means the singular homology.*

The following remark shows that we cannot take $k = 1$ and $k = \infty$ in Theorem 1.1.

Remark 1.2. (1) Let $n \in \mathbb{N}$ and Y be a one-point set. Then the constant map from $\mathbb{R}^n$ to Y is a continuous map and $\mathbb{R}^n$ and Y satisfy the conditions on Theorem 1.1.

(2) Let $n \in \mathbb{N}$. Then $\mathbb{R}^n$ has the free $\mathbb{Z}$-action defined by $\mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$, $(g, x_1, ..., x_n) \mapsto (g + x_1, x_2, ..., x_n)$. Therefore, $\mathbb{R}^n$ and $\mathbb{R}$ satisfy the assumptions on Theorem 1.1 and the map $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x_1, ..., x_n) = x_1$ is a continuous $\mathbb{Z}$-map.

Let k be a prime. For a topological space Y, let $D = \{(y_1, ..., y_k) \in Y \times \cdots \times Y \mid y_1 = \cdots = y_k\}$ be the diagonal and write $Y^* = Y \times \cdots \times Y - D$ admitting the free C_k-action defined by $g(y_1, y_2, ..., y_k) = (y_2, y_3, ..., y_k, y_1)$, where g generates C_k.

Theorem 1.3. *Let k be a prime and X be an arcwise connected free C_k-space. If there exists a positive integer n such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \le q \le n$ and Y is a Hausdorff space with $H_{n+1}(Y^*/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then every continuous map $f : X \to Y$ has a C_k-coincidence point, that is, a point x such that $f(x) = f(gx) = \cdots = f(g^{k-1}x)$, where g is a generator of C_k.*

We can consider the definable versions of Theorem 1.1 and Theorem 1.3 in an o-minimal expansion $\mathcal{N} = (R, +, \cdot, <, ...)$ of a real closed field R.

Many results in the semialgebraic geometry hold in the o-minimal setting and there exist uncountably many o-minimal expansions of the standard structure of the field $\mathbb{R}$ of real numbers ([19]). See also [4], [6], [12] for examples and constructions of o-minimal structures. General references on them are [3], [5], [20]. In this paper, "definable" means "definable with parameters in $\mathcal{N}$", every definable object is considered in $\mathcal{N}$ and each definable map is continuous unless otherwise stated.

Let S^n denote the n-dimensional unit sphere of R^{n+1}. If $R = \mathbb{R}_{alg}$, then S^n is neither arcwise connected nor connected. Thus we cannot apply [21], [18], [11], [7] and Theorem 1.1 even if $X = S^{2n+1}$ and $Y = S^{2m+1}$.

The singular definable homology is introduced in [22]. Using the singular definable homology, we have the following theorem which is a definable version of Theorem 1.1.

Theorem 1.4. *Let X be a definably connected definable set with a free definable C_k-action. If there exists a positive integer n such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \le q \le n$ and Y is a definable set with a free definable C_k-action such that $H_{n+1}(Y/C_k\,; \mathbb{Z}/k\mathbb{Z}) = 0$, then there is no definable C_k-map from X to Y. Here this homology means the singular definable homology.*

Note that a definably connected definable set is not necessarily connected and a definable set is definably connected if and only if definably arcwise connected. Here a definable set X is definably arcwise connected if for every two points $x,\ y \in X$, there exists a definable map f from the closed unit interval $[0, 1]_R$ of R to X such that $x = f(0)$ and $y = f(1)$.

In the definable setting, we have the following simple sufficient condition on Y which implies $H_{n+1}(Y/C_k\,; \mathbb{Z}/k\mathbb{Z}) = 0$.

If Y is a definable set with a definable C_k-action, then by Corollary 10.2.18 in [3], Y/C_k is a definable set and the orbit map $\pi : Y \to Y/C_k$ is definable. If dim $Y \le n$, then by Corollary 4.1.6 in [3], $\dim Y/C_r \le n$. Thus if $\dim Y \le n$, then $H_{n+1}(Y/C_k\,; \mathbb{Z}/k\mathbb{Z}) = 0$.

Corollary 1.5. (1) *Suppose that $k \ge 3$ and that C_k acts on S^{2m+1} and S^{2n+1} definably and freely. If there exists a definable C_k-map $f : S^{2m+1} \to S^{2n+1}$, then $m \le n$.*

(2) *If S^m and S^n have free definable C_2-actions and there exists a definable C_2-map $f : S^m \to S^n$, then $m \le n$.*

Corollary 1.5 is a generalization of Theorem 1.1 [16].

Using Theorem 1.4, we have the following theorem.

Theorem 1.6. *Let k be a prime and X be a definably connected definable set with a free definable C_k-action. Assume that there exists a positive integer n such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \le q \le n$. If Y is a definable set with $H_{n+1}(Y^*/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then every definable map $f : X \to Y$ has a C_k-coincidence point, that is, a point x such that $f(x) = f(gx) = \cdots = f(g^{k-1}x)$, where g is a generator of C_k.*

2. Proof of Theorem 1.1 and Theorem 1.3

We first prove Theorem 1.1.

Let $\mathbb{Z}/k\mathbb{Z}[C_k]$ denote the group ring of C_k over $\mathbb{Z}/k\mathbb{Z}$. For any $q \in \mathbb{N} \cup \{0\}$, the q-dimensional chain group $C_q(X; \mathbb{Z}/k\mathbb{Z})$ has the standard C_k-action. Then this action induces $\mathbb{Z}/k\mathbb{Z}[C_k]$-action on $C_q(X; \mathbb{Z}/k\mathbb{Z})$.

Let g be a generator of C_k, $\alpha = 1 + g + \cdots + g^{k-1}$, and $\beta = 1 - g$. Then by definition $\alpha\beta = \beta\alpha = 0$, for every q, $\alpha C_q(X; \mathbb{Z}/k\mathbb{Z})$ and $\beta C_q(X; \mathbb{Z}/k\mathbb{Z})$ are $\mathbb{Z}/k\mathbb{Z}[C_k]$-submodules of $C_q(X; \mathbb{Z}/k\mathbb{Z})$ and $\alpha\partial = \partial\alpha$, $\beta\partial = \partial\beta$, where ∂ is the boundary operator of $\{C_q(X; \mathbb{Z}/k\mathbb{Z})\}$. Therefore, $\{\alpha C_q(X; \mathbb{Z}/k\mathbb{Z})\}$ and $\{\beta C_q(X; \mathbb{Z}/k\mathbb{Z})\}$ are subchain complexes of $\{C_q(X; \mathbb{Z}/k\mathbb{Z})\}$.

Proposition 2.1. *For every q, the following two sequences are exact:*

$$0 \to \alpha C_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{i} C_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\beta} \beta C_q(X; \mathbb{Z}/k\mathbb{Z}) \to 0,$$

$$0 \to \beta C_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{j} C_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\alpha} \alpha C_q(X; \mathbb{Z}/k\mathbb{Z}) \to 0,$$

where i, j denote the inclusions and α (resp. β) stands for the multiplication of α (resp. β).

Proof. Since $\beta \circ i = 0$, $\alpha \circ j = 0$, $\operatorname{Im} i \subset \operatorname{Ker}\beta$, $\operatorname{Im} j \subset \operatorname{Ker}\alpha$.

Let $s = \sum_j \sum_{i=0}^{k-1} n_{ji} g^i \sigma_j \in \operatorname{Ker}\beta$, where g is a generator of C_k. If $l \ne l'$

and $0 \leq i \leq k-1$, then $g^i\sigma_l \neq \sigma_{l'}$. Since $\beta s = 0$, for any j, $\sum_{i=0}^{k-1} n_{ji} g^i (1-g)\sigma_j = 0$. Thus for every j, $\sum_{i=1}^{k-1}(n_{ji} - n_{j(i-1)})g^i\sigma_i + (n_{j0} - n_{j(k-1)})\sigma_j = 0$. Hence for each j, $n_{j0} = n_{j1} = \cdots = n_{jk-1}$. We set $n_j = n_{j0} (= n_{j1} = \cdots = n_{jk-1})$. Then we have $s = \sum_j n_j(1 + g + \cdots + g^{k-1})\sigma_j = \alpha \sum_j n_j \sigma_j \in \operatorname{Im} i$. Therefore, $\operatorname{Ker}\beta = \operatorname{Im} i$.

Let $s = \sum_j \sum_{i=0}^{k-1} n_{ji} g^i \sigma_j \in \operatorname{Ker}\alpha$. Since $\alpha s = \sum_j (n_{j0} + \cdots + n_{j(k-1)})(1 + \cdots + g^{k-1})\sigma_j = 0$, $n_{j0} + \cdots + n_{j(k-1)} = 0$.

Thus $s = \sum_j (n_{j0}(1-g) + (n_{j0} + n_{j1})g(1-g) + (n_{j0} + n_{j1} + n_{j2})g^2(1-g) + \cdots + (n_{j0} + n_{j1} + \cdots + n_{j(k-2)})g^{k-2}(1-g))\sigma_j \in \operatorname{Im} j$. Therefore, $\operatorname{Ker}\alpha = \operatorname{Im} j$. □

Let $H_q^\alpha(X, \mathbb{Z}/k\mathbb{Z})$ (resp. $H_q^\beta(X; \mathbb{Z}/k\mathbb{Z})$) denote the homology group induced from the chain complex $\{\alpha C_q(X; \mathbb{Z}/k\mathbb{Z})\}$ (resp. $\{\beta C_q(X; \mathbb{Z}/k\mathbb{Z})\}$). We call these homology groups the *Smith homology groups*.

By Proposition 2.1, we have the following theorem.

Theorem 2.2. *The following two sequences are exact:*

$$\cdots \to H_q^\alpha(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{i_*} H_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\beta_*} H_q^\beta(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\partial_*} H_{q-1}^\alpha(X; \mathbb{Z}/k\mathbb{Z}) \to \cdots$$

$$\cdots \to H_q^\beta(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{j_*} H_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\alpha_*} H_q^\alpha(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\partial'_*} H_{q-1}^\beta(X; \mathbb{Z}/k\mathbb{Z}) \to \cdots.$$

In particular, if $k = 2$, then $\alpha = \beta$ and

$$\cdots \to H_q^\alpha(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i_*} H_q(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\alpha_*} H_q^\alpha(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial_*} H_{q-1}^\alpha(X; \mathbb{Z}/2\mathbb{Z}) \to \cdots$$

is exact.

Let $p : E \to X$ be a continuous map. We say that p has the *homotopy lifting property* if for any compact space Z, each homotopy $h : Z \times [0, 1] \to X$ and a continuous map $F : Z \to E$ such that $p \circ F(z) = h(z, 0)$, for all $z \in Z$, there exists a homotopy $H : Z \times [0, 1] \to E$ such that $p \circ H = h$ and $H(z, 0) = F(z)$, for all $z \in Z$, where $[0, 1]$ denotes the closed unit interval of $\mathbb{R}$.

Proposition 2.3. *Let X be an arcwise connected Hausdorff free C_k-space. Then the orbit map $\pi : X \to X/C_k$ has the homotopy lifting property.*

Proposition 2.4. *If Y is an arcwise connected Hausdorff free C_k-space, then for every q,* $H_q^{\alpha}(Y, \mathbb{Z}/k\mathbb{Z}) \cong H_q(Y/C_k, \mathbb{Z}/k\mathbb{Z})$.

Proof. We first show that the map $\alpha : C(Y; \mathbb{Z}/k\mathbb{Z}) \to C(Y; \mathbb{Z}/k\mathbb{Z})$ and the map $\pi_* : C(Y; \mathbb{Z}/k\mathbb{Z}) \to C(Y/C_k; \mathbb{Z}/k\mathbb{Z})$ induced from the orbit map $\pi : Y \to Y/C_k$ have the same kernel. Let σ be a singular s-simplex of Y. We need only to consider elements of $C(C_k\sigma)$, since $C(Y) \cong \oplus_{[\sigma]\in\Delta(s)/C_k} C(C_k\sigma)$, where $\Delta(s)$ is the set of singular s-simplexes of Y and $\Delta(s)/C_k$ is its orbit set under the induced action.

Since $\alpha\left(\sum n_i g^i \sigma\right) = \left(\sum n_i\right)\alpha(\sigma)$, $\alpha\left(\sum n_i g^i \sigma\right) = 0$ if and only if $\sum n_i = 0$, and similarly $\pi_*\left(\sum n_i g^i \sigma\right) = \left(\sum n_i\right)\pi \circ \sigma = 0$ if and only if $\sum n_i = 0$; therefore, both kernels coincide.

We next show that π_* is surjective; namely, there is a lift $\tilde{\tau} : \Delta^s \to Y$ of $\tau : \Delta^s \to Y/C_k$, where Δ^s denotes the affine span of $(s+1)$-points which are affine independent. Since Δ^s is contractible, there is a homotopy $H' : \Delta^s \times [0, 1] \to \Delta^s$ such that $H'(-, 0) = c_{e_0}$ and $H'(-, 1) = id_{\Delta^s}$, where c_{e_0} denotes the constant map whose value is $e_0 \in \Delta^s$. Then the composition $H = \tau \circ H'$ is a homotopy from the constant map $c_{\tau(e_0)}$ to τ. Let y_0 be a point of Y such that $\pi(y_0) = \tau(e_0)$, and $c_{y_0} : \Delta^s \to Y$ the constant map whose value is y_0. Since $H(-, 0) = \pi \circ c_{y_0}$, it follows from Proposition 2.3 that there exists a lift $\tilde{H} : \Delta^s \times [0, 1] \to Y$ of H such that $\tilde{H}(-, 0) = c_{y_0}$. Then $\tilde{\tau} := \tilde{H}(-, 1)$ is a lift of $\tau = H(-, 1)$.

Since π_* is surjective, $\alpha C(Y; \mathbb{Z}/k\mathbb{Z})$ and $C(Y/C_p; \mathbb{Z}/k\mathbb{Z})$ are isomorphic as chain complexes. Accordingly their homology groups are also isomorphic. □

Proof of Theorem 1.1. Assume that there exists a continuous C_k-map $f : X$

$\to Y$ under the conditions of Theorem 1.1. Since X is arcwise connected, $f(X)$ is acrwise connected. Hence $f(X)$ is contained in an arcwise connected component of Y. Therefore it is sufficient to prove the case where Y is arcwise connected.

We first prove the case where $k = 2$. Since f is a continuous C_2-map, $\alpha f_{\#} = f_{\#}\alpha$.

For simplicity, we abbreviate the coefficient $\mathbb{Z}/2\mathbb{Z}$ in the singular homology. By Theorem 2.2, we have a commutative diagram

$$\begin{array}{ccccccccccccc}
\to & H^{\alpha}_{n+1}(X) & \xrightarrow{\partial^X_*} & H^{\alpha}_n(X) & \xrightarrow{i^X_*} & H_n(X) & \xrightarrow{\alpha^X_*} & H^{\alpha}_n(X) & \xrightarrow{\partial^X_*} & H^{\alpha}_{n-1}(X) & \to & \cdots \\
 & f^{\alpha}_* \downarrow & & f^{\alpha}_* \downarrow & & f_* \downarrow & & f^{\alpha}_* \downarrow & & f^{\alpha}_* \downarrow & & \\
\to & H^{\alpha}_{n+1}(Y) & \xrightarrow{\partial^Y_*} & H^{\alpha}_n(Y) & \xrightarrow{i^Y_*} & H_n(Y) & \xrightarrow{\alpha^Y_*} & H^{\beta}_n(Y) & \xrightarrow{\partial^Y_*} & H^{\alpha}_{n-1}(Y) & \to & \cdots
\end{array}$$

$$\begin{array}{ccccccccccccccc}
\to & H^{\alpha}_1(X) & \xrightarrow{i^X_*} & H_1(X) & \xrightarrow{\alpha^X_*} & H^{\alpha}_1(X) & \xrightarrow{\partial^X_*} & H^{\alpha}_0(X) & \xrightarrow{i^X_*} & H_0(X) & \xrightarrow{\alpha^X_*} & H^{\alpha}_0(X) & \to & 0 \\
 & f^{\alpha}_* \downarrow & & f_* \downarrow & & f^{\alpha}_* \downarrow & & f^{\alpha}_* \downarrow & & f_* \downarrow & & f^{\alpha}_* \downarrow & & f_* \downarrow \\
\to & H^{\alpha}_1(Y) & \xrightarrow{i^Y_*} & H_1(Y) & \xrightarrow{\alpha^Y_*} & H^{\alpha}_1(Y) & \xrightarrow{\partial^Y_*} & H^{\alpha}_0(Y) & \xrightarrow{i^Y_*} & H_0(Y) & \xrightarrow{\alpha^Y_*} & H^{\alpha}_0(Y) & \to & 0
\end{array}$$

with exact rows.

By definition, $(i^X_*)_0 = 0$ and $(i^Y_*)_0 = 0$. Thus $(\alpha^X_*)_0 : H_0(X) \to H^{\alpha}_0(X)$ and $(\alpha^Y_*)_0 : H_0(Y) \to H^{\alpha}_0(Y)$ are isomorphisms. By assumption, $H_0(X) \cong \mathbb{Z}/2\mathbb{Z}$. Hence $H_0(X) \cong H^{\alpha}_0(X) \cong \mathbb{Z}/2\mathbb{Z}$. Similarly, $H_0(Y) \cong H^{\alpha}_0(Y) \cong \mathbb{Z}/2\mathbb{Z}$. Since $(f_*)_0 : H_0(X) \to H_0(Y)$ is an isomorphism and $(\alpha^X_*)_0 \circ (f_*)_0 = (f^{\alpha}_*)_0 \circ (\alpha^X_*)_0$, $(f^{\alpha}_*)_0 : H^{\alpha}_0(X) \to H^{\alpha}_0(Y)$ is an isomorphism. Since $(i^X_*)_0 = 0$, we have $\mathrm{Im}(\partial^X_*)_1 = \mathrm{Ker}\,(i^X_*)_0 = H^{\alpha}_0(X)$. Thus we see that $(\partial^Y_*)_1 \circ (f^{\alpha}_*)_1 = (f^{\alpha}_*)_0 \circ (\partial^X_*)_1 :$ $H^{\alpha}_1(X) \to H^{\alpha}_0(Y)$ is a non-zero homomorphism. Hence $(f^{\alpha}_*)_1 : H^{\alpha}_1(X) \to H^{\alpha}_1(Y)$ is a non-zero homomorphism. Using the assumptions on X, we see that $(\partial^X_*)_q :$ $H^{\alpha}_q(X) \to H^{\alpha}_{q-1}(X)$ is an isomorphism for each $1 \le q \le n$. Using this fact and by induction, we can prove that $(f^{\alpha}_*)_q : H^{\alpha}_q(X) \to H^{\alpha}_q(Y)$ is a non-zero homomorphism for each $0 \le q \le n$.

By Proposition 2.4, $H^{\alpha}_{n+1}(Y) \cong H_{n+1}(Y/C_p)$. Thus $H^{\alpha}_{n+1}(Y) = 0$. Hence $(i^Y_*)_n : H^{\alpha}_n(Y) \to H_n(Y)$ is injective and $(i^X_*)_n \circ (f^{\alpha}_*)_n : H^{\alpha}_n(X) \to H_n(Y)$ is a non-zero homomorphism.

On the other hand, since $H_n(X) = 0$, $(i^Y_*)_n \circ (f^{\alpha}_*)_n = (f_*)_n \circ (i^X_*)_n = 0$. This contradiction proves the theorem in this case.

Next, we prove the case where $k > 2$. For simplicity, we abbreviate the coefficient $\mathbb{Z}/k\mathbb{Z}$ in the singular homology. By Theorem 2.2, we have two commutative diagrams

$$\begin{array}{ccccccccc}
\to & H^{\alpha}_n(X) & \xrightarrow{i^X_*} & H_n(X) & \xrightarrow{\beta^X_*} & H^{\beta}_n(X) & \xrightarrow{\partial^X_*} & H^{\alpha}_{n-1}(X) & \to \cdots \\
 & f^{\alpha}_* \downarrow & & f_* \downarrow & & f^{\beta}_* \downarrow & & f^{\alpha}_* \downarrow & \\
\to & H^{\alpha}_n(Y) & \xrightarrow{i^Y_*} & H_n(Y) & \xrightarrow{\beta^Y_*} & H^{\beta}_n(Y) & \xrightarrow{\partial^Y_*} & H^{\alpha}_{n-1}(Y) & \to \cdots
\end{array}$$

$$\begin{array}{ccccccccccccccc}
\to & H^{\alpha}_1(X) & \xrightarrow{i^X_*} & H_1(X) & \xrightarrow{\beta^X_*} & H^{\beta}_1(X) & \xrightarrow{\partial^X_*} & H^{\alpha}_0(X) & \xrightarrow{i^X_*} & H_0(X) & \xrightarrow{\beta^X_*} & H^{\beta}_0(X) & \to & 0 \\
 & f^{\alpha}_* \downarrow & & f_* \downarrow & & f^{\beta}_* \downarrow & & f^{\alpha}_* \downarrow & & f_* \downarrow & & f^{\beta}_* \downarrow & & f_* \downarrow \\
\to & H^{\alpha}_1(Y) & \xrightarrow{i^Y_*} & H_1(Y) & \xrightarrow{\beta^Y_*} & H^{\beta}_1(Y) & \xrightarrow{\partial^Y_*} & H^{\alpha}_0(Y) & \xrightarrow{i^Y_*} & H_0(Y) & \xrightarrow{\beta^Y_*} & H^{\beta}_0(Y) & \to & 0
\end{array}$$

and

$$\begin{array}{ccccccccccc}
\to & H^{\alpha}_{n+1}(X) & \xrightarrow{\partial'^X_*} & H^{\beta}_n(X) & \xrightarrow{j^X_*} & H_n(X) & \xrightarrow{\alpha^X_*} & H^{\alpha}_n(X) & \xrightarrow{\partial'^X_*} & H^{\beta}_{n-1}(X) & \to \cdots \\
 & f^{\alpha}_* \downarrow & & f^{\beta}_* \downarrow & & f_* \downarrow & & f^{\alpha}_* \downarrow & & f^{\beta}_* \downarrow & \\
\to & H^{\alpha}_{n+1}(Y) & \xrightarrow{\partial'^Y_*} & H^{\beta}_n(Y) & \xrightarrow{j^Y_*} & H_n(Y) & \xrightarrow{\alpha^Y_*} & H^{\alpha}_n(Y) & \xrightarrow{\partial'^Y_*} & H^{\beta}_{n-1}(Y) & \to \cdots
\end{array}$$

$$\begin{array}{ccccccccccccccc}
\to & H^{\beta}_1(X) & \xrightarrow{j^X_*} & H_1(X) & \xrightarrow{\alpha^X_*} & H^{\alpha}_1(X) & \xrightarrow{\partial'^X_*} & H^{\beta}_0(X) & \xrightarrow{j^X_*} & H_0(X) & \xrightarrow{\alpha^X_*} & H^{\alpha}_0(X) & \to & 0 \\
 & f^{\beta}_* \downarrow & & f_* \downarrow & & f^{\alpha}_* \downarrow & & f^{\beta}_* \downarrow & & f_* \downarrow & & f^{\alpha}_* \downarrow & & f_* \downarrow \\
\to & H^{\beta}_1(Y) & \xrightarrow{j^Y_*} & H_1(Y) & \xrightarrow{\alpha^Y_*} & H^{\alpha}_1(Y) & \xrightarrow{\partial'^Y_*} & H^{\beta}_0(Y) & \xrightarrow{j^Y_*} & H_0(Y) & \xrightarrow{\alpha^Y_*} & H^{\alpha}_0(Y) & \to & 0
\end{array}$$

with exact rows.

We easily see that $(i^X_*)_0 = 0$ and $(i^Y_*)_0 = 0$. Thus $(\beta^X_*)_0 : H_0(X) \to H^{\beta}_0(X)$ and $(\beta^Y_*)_0 : H_0(Y) \to H^{\beta}_0(Y)$ are isomorphisms. Since $(f_*)_0 : H_0(X) \to H_0(Y)$ is an isomorphism, $(f^{\beta}_*)_0 : H^{\beta}_0(X) \to H^{\beta}_0(Y)$ is an isomorphism. Similarly, we see that $(f^{\alpha}_*)_0 : H^{\alpha}_0(X) \to H^{\alpha}_0(Y)$ is an isomorphism from the second diagram. Since $H_1(X) = 0$ and $(i^X_*)_0 = 0$, $(\partial^X_*)_1 : H^{\beta}_1(X) \to H^{\alpha}_0(X)$ is an isomorphism.