(2) *$G$ has the $IB$-property in $\mathcal{S}_G$.*

(3) *$G$ is a solvable compact Lie group.*

*Proof.* We have already seen the implication (3) ⇒ (1), and trivially (1) ⇒ (2). To see (2) ⇒ (3), we show that there is a counterexample to the isovariant Borsuk-Ulam theorem when $G$ is nonsolvable. According to Oliver [40, Theorem 4], there exists a disk $D$ with a smooth $G$-action such that $D^H$ is also a disk if $H$ is a solvable subgroup, and the empty set if $H$ is a nonsolvable subgroup. The boundary of this $G$-disk $D$ is clearly a semilinear $G$-sphere. Note also that $D^G = \emptyset$ since $G$ is nonsolvable. Set $\Sigma_n = \partial(D \times D(\mathbb{R}^n))$, where $D(\mathbb{R}^n)$ is an $n$-dimensional disk with trivial $G$-action. Then $\Sigma_n$ is a semilinear $G$-sphere without $G$-fixed points. For any positive integer $n$, take a map $h_n : D(\mathbb{R}^n) \to D(\mathbb{R}^1)$ such that $h_n(D(\mathbb{R}^n)) \subset \partial D(\mathbb{R}^1)$, and define a $G$-map

$$g_n := id \times h_n : D \times D(\mathbb{R}^n) \to D \times D(\mathbb{R}^1).$$

Then one can easily see that $g_n$ is a $G$-isovariant map and $g_n$ maps the boundary $\partial(D \times D(\mathbb{R}^n))$ into the boundary $\partial(D \times D(\mathbb{R}^1))$. Hence we obtain a $G$-isovariant map $f_n := g_{n|\Sigma_n} : \Sigma_n \to \Sigma_1$. Since $\dim \Sigma_n > \dim \Sigma_1$ for $n > 1$, this $f_n$ gives a counterexample to the isovariant Borsuk-Ulam theorem. □

*Remark.* The semilinear $G$-sphere $\Sigma_1$ in the above proof is equivariantly embedded in some linear $G$-sphere $SW$ [6]. Hence there is an isovariant map $f_n : \Sigma_n \to SW$ such that $\dim \Sigma_n + 1 > \dim SW - \dim SW^G$ for some large $n$.

In the case $\mathcal{F}_G = \mathcal{L}_G$, the problem is more difficult; in fact, a complete answer is unfortunately unknown to the best of the author's knowledge. However some partial answers are known. We present them here without proof.

To state Wasserman's result, we recall the *prime condition* for a finite group $G$.

**Definition.** We say that a finite simple group $G$ satisfies the prime condition if for every element $g \in G$,

$$\sum_{p|o(g)} \frac{1}{p} \leq 1$$

holds, where $o(g)$ denotes the order of $g$, and $p$ is a prime dividing $o(g)$.

We say that a finite group $G$ satisfies the prime condition if every simple factor group in a normal series of $G$ satisfies the prime condition as a simple group.

**Theorem 3.2** ([49]). *Every finite group satisfying the prime condition has the IB-property in $\mathcal{L}_G$.*

This theorem provides nonsolvable examples of having the IB-property in $\mathcal{L}_G$.

**Example 3.3.** The alternating groups $A_5, A_6, \ldots, A_{11}$ satisfy the prime condition, and hence $A_i$ has the IB-property in $\mathcal{L}_{A_i}$, $i = 5, 6, \ldots, 11$.

*Remark.* The alternating groups $A_n$, $n > 11$, do not satisfy the prime condition. However, the author does not know whether $A_n$ has the IB-property for $n > 11$.

Another partial answer is a weak version of the isovariant Borsuk-Ulam theorem.

**Theorem 3.4** ([31]). *For an arbitrary compact Lie group $G$, there exists a weakly monotone increasing function $\varphi_G : \mathbb{N}_0 \to \mathbb{N}_0$ diverging to $\infty$ with the following property.*

(WIB) *For any pair of representations $V$ and $W$ such that there is a $G$-isovariant map $f : SV \to SW$, the inequality*

$$\varphi_G(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

*holds.*

*Here $\mathbb{N}_0$ denotes the set of nonnegative integers.*

The above result does not hold for $G$-equivariant maps even if $SV^G = SW^G = \emptyset$. For example, when $G$ is a cyclic group $C_{pq}$ of order $pq$, where $p$, $q$ are distinct primes, a Borsuk-Ulam type theorem does not hold as can be seen below.

Let $U_k$ $(= \mathbb{C})$ be the representation of $C_n = \langle g \rangle$ for which $g$ acts by $g \cdot z = \xi^k z$, $z \in U_k$, where $\xi = \exp(2\pi\sqrt{-1}/n)$.

**Proposition 3.5** (cf. [48]). *Let $C_{pq} = \langle g \rangle$ be a cyclic group of order $pq$, where $p$, $q$ are distinct primes. For any positive integer $r$, there is a $C_{pq}$-map $f : S(rU_1) \to S(U_p \oplus U_q)$, where $rU_1$ is the direct sum of $r$ copies of $U_1$.*

*Proof.* Set $G = C_{pq}$. By a result of [48], there is a self $G$-map $h : S(U_p \oplus U_q) \to S(U_p \oplus U_q)$ with $\deg h = 0$; hence $h$ is (nonequivariantly) nullhomotopic. Since $G$ acts freely on $S(rU_1)$, $S(rU_1)$ has a $G$-CW complex structure consisting of free $G$-cells. We put $S(rU_1) = \bigcup_k X_k$, where $X_k$ is the $k$-skeleton. A $G$-map from $S(rU_1)$

to $S(U_p \oplus U_q)$ is inductively constructed as follows. Suppose that one has a $G$-map $f_{k-1} : X_{k-1} \to S(U_p \oplus U_q)$. Then $f_{k-1}$ can be extended to a $G$-map $f' : X_{k-1} \cup_\phi G \times D^k \to S(U_p \oplus U_q)$. Indeed, since $h \circ \phi_{|\{1\} \times S^{k-1}} : S^{k-1} \to S(U_p \oplus U_q)$ is nullhomotopic, $h \circ \phi_{|S^{k-1}}$ is extended to a map $g : D^k \to S(U_p \oplus U_q)$, and next $g$ is equivariantly extended to a $G$-map $g' : G \times D^k \to S(U_p \oplus U_q)$. By gluing $g'$ to $f_{k-1}$, one obtains a $G$-map $f' : X_{k-1} \cup_\phi G \times D^k \to S(U_p \oplus U_q)$. Repeating this procedure, one has a $G$-map $f_k : X_k \to S(U_p \oplus U_q)$. □

*Remark.* More generally, Bartsch [1] shows that a weak version of the Borsuk-Ulam theorem holds for linear $G$-spheres of a finite group $G$ if and only if $G$ has prime power order.

Combining this proposition with the isovariant Borsuk-Ulam theorem, we obtain another Borsuk-Ulam type result.

**Corollary 3.6.** *For any $C_{pq}$-map $f : S(rU_1) \to S(U_p \oplus U_q)$, $r \geq 2$, the image of $f$ meets the Hopf link $SU_p \coprod SU_q$ in $S(U_p \oplus U_q)$.*

*Proof.* Suppose that $f^{-1}(SU_p \coprod SU_q) = \emptyset$. Then $f$ is a $G$-isovariant map, since $G$ acts freely on $S(U_p \oplus U_q) \setminus (SU_p \coprod SU_q)$. Furthermore $\mathrm{Res}_{C_p} f$ is a $C_p$-isovariant map. By the isovariant Borsuk-Ulam theorem, it follows that

$$2r = \dim S(rU_q) + 1 \leq \dim S(U_p \oplus U_q) - \dim S(U_p \oplus U_q)^{C_p} = 2.$$

This is a contradiction. □

*Remark.* Another equivalent statement of the original Borsuk-Ulam theorem is: For any $C_2$-map $f : S^n \to \mathbb{R}^n$, the image of $f$ meets the origin in $\mathbb{R}^n$, where the $C_2$-actions on $S^n$ and $\mathbb{R}^n$ are both given by multiplication by $-1$.

# 4 The converse of the isovariant Borsuk-Ulam theorem

The isovariant Borsuk-Ulam theorem is interpreted as a nonexistence result of isovariant maps, and it produces several inequalities, which give a necessary condition for the existence of an isovariant map. In several cases, it is also sufficient. In this section, we shall present such examples for the existence of an isovariant map and discuss the converse of the isovariant Borsuk-Ulam theorem. The materials are taken from [34], [35], [38], [39].

As a special case of Corollary 2.8, we consider the case $N = SW$, a linear $G$-sphere. Then, the inequality

$$\dim M + 1 \leq \dim SW - \dim SW^{>1}$$

holds if there is a $G$-isovariant map $f : M \to SW$. In this case, we show that the converse is true.

**Proposition 4.1.** *Let $G$ be a finite group and $M$ a compact $G$-manifold with free $G$-action. Let $W$ be a representation of $G$. If $\dim M + 1 \leq \dim SW - \dim SW^{>1}$, then there exists a $G$-isovariant map $f : M \to SW$.*

We define the free part $SW_{\mathrm{free}}$ of $SW$ by $SW_{\mathrm{free}} = SW \setminus SW^{>1}$. Set $d = \dim SW - \dim SW^{>1}$.

**Lemma 4.2.** *The free part $SW_{\mathrm{free}}$ is $(d-2)$-connected.*

*Proof.* Since $\dim S^k \times I + \dim SW^{>1} < \dim SW$ for $k \leq d-2$, any homotopy into $SW$ deforms to a homotopy into $SW_{\mathrm{free}}$ by a general position argument. Hence every map from $S^k$ to $SW_{\mathrm{free}}$ is nullhomotopic for $k \leq d-2$. □

*Proof of Proposition* 4.1. Since $G$ acts freely on $M$ and $SW_{\mathrm{free}}$, it suffices to show the existence of a $G$-map from $M$ to $SW_{\mathrm{free}}$. Since $M$ has a $G$-CW complex structure, we may put $M = \bigcup_k X_k$, where $X_k$ is the $k$-skeleton of $M$. The inequality means $k \leq d-1$. A $G$-map into $SW_{\mathrm{free}}$ is inductively constructed as follows. Suppose that $f_{k-1}$ is constructed as a $G$-map from $X_{k-1}$ to $SW_{\mathrm{free}}$; then $f_{k-1}$ is extended on $X_{(k-1)} \cup_\phi G \times D^k$; indeed, since $SW_{\mathrm{free}}$ is $(d-2)$-connected, the map $f_{k-1} \circ \phi_{|\{1\} \times S^{k-1}} : S^{k-1} \to SW_{\mathrm{free}}$ is extended to a map $g : D^k \to SW_{\mathrm{free}}$ and then $g$ is equivariantly extended to a $G$-map $\tilde{g} : G \times D^k \to SW_{\mathrm{free}}$. By gluing $\tilde{g}$ to $f_{k-1}$, one can obtain a $G$-map $f'_{k-1} : X_{k-1} \cup_\phi G \times D^k \to SW_{\mathrm{free}}$. Repeating this procedure, we obtain a $G$-map from the $k$-skeleton $X_k$ to $SW_{\mathrm{free}}$, and finally we obtain a $G$-map $f : M \to SW_{\mathrm{free}}$. □

Thus, in this situation, the existence problem is solved as follows.

**Corollary 4.3.** *Let $M$ be a mod $|G|$ homology sphere with free $G$-action and $W$ a representation of $G$. There exists a $G$-isovariant map from $M$ to $SW$ if and only if*

$$\dim M + 1 \leq \dim SW - \dim SW^{>1}.$$

Next we consider the existence problem of an isovariant map between (real) representations. Let $G$ be a finite solvable group. Let $f : V \to W$ be a $G$-isovariant map between representations $V$ and $W$. Take any pair $(H, K)$ of subgroups of $G$ with $H \lhd K$. Then $f^H : V^H \to W^H$ is considered as a $K/H$-isovariant map. Since $K/H$ is solvable, the isovariant Borsuk-Ulam theorem implies the inequality

$$\dim V^H - \dim V^K \leq \dim W^H - \dim W^K.$$

From this observation, we consider the following condition for a pair of representations $V$ and $W$ of a solvable group $G$:

$(C_{V,W})$ $\dim V^H - \dim V^K \leq \dim W^H - \dim W^K$ for every pair $(H, K)$ with $H \lhd K$.

Moreover the condition $(I_{V,W})$: $\operatorname{Iso} V \subset \operatorname{Iso} W$ is obviously necessary for the existence of an isovariant map.

**Definition.** We say that a finite solvable group $G$ has the *complete IB-property* if, for every pair $(V, W)$ of representations satisfying conditions $(C_{V,W})$ and $(I_{V,W})$, there exists a $G$-isovariant map from $V$ to $W$.

*Remark.* If $G$ is nilpotent, then $(C_{V,W})$ implies $(I_{V,W})$ [34].

Which solvable groups have the complete IB-property? A complete answer is not known; however, some examples that have the complete IB-property are known.

**Theorem 4.4** ([34], [35]). *Let $p$, $q$, $r$ be distinct primes. The following finite groups have the complete IB-property:*

(1) *abelian $p$-groups,*

(2) $C_{p^m q^n}$*: cyclic groups of order* $p^m q^n$,

(3) $C_{pqr}$*: cyclic groups of order* $pqr$,

(4) $D_3$, $D_4$ *and* $D_6$*: dihedral groups of order* 6, 8 *and* 12, *respectively.*

*Remark.* S. Kôno also obtains a similar result in the case of complex $C_{p^m q^n}$-representations.

For details of the proof, see [34] and [35]. The idea is to decompose $(V, W)$ into primitive pairs $(V_i, W_i)$.

**Definition.** A pair of representations $(V, W)$ is called *primitive* if $V$ and $W$ cannot be decomposed into $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$ such that $(V_i, W_i)$, $V_i \neq 0$, $W_i \neq 0$, satisfies $(C_{V_i,W_i})$, $i = 1, 2$.

Then, by constructing a $G$-isovariant map $f_i : V_i \to W_i$, we have a $G$-isovariant map $f = \oplus_i f_i : V \to W$.

**Example 4.5.** The following are examples of primitive pairs of $C_n$-representations, and there exist isovariant maps between the representations. Suppose that $p$, $q$, $r$ are pairwise coprime integers greater than 1.

(1) $(U_k, U_l)$ when $(k, n) = (l, n) = 1$.

(2) $(U_1, U_p \oplus U_q)$ when $pq$ divides $n$.

(3) $(U_p \oplus U_q,\ U_{p^2} \oplus U_{pq})$ when $p^2 q$ divides $n$.

(4) $(U_p \oplus U_q \oplus U_r,\ U_1 \oplus U_{pq} \oplus U_{qr} \oplus U_{pr})$ when $pqr$ divides $n$.

In the cases (1)-(3), one can define a $C_n$-isovariant map concretely; however, in case (4), equivariant obstruction theory is used. We illustrate it in Section 5.

On the other hand, there exists a group not having the complete IB-property.

**Theorem 4.6** ([35])**.** *Let $D_n$ be the dihedral group of order $2n$ $(n \geq 3)$. Every $D_n$ $(n \neq 3, 4, 6)$ does not have the complete IB-property.*

The dihedral group $D_n$ has the following presentation:

$$D_n = \langle a, b \,|\, a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle.$$

One has the normal cyclic subgroup $C_m = \langle a^{n/m} \rangle$ of $D_n$ for every divisor $m$ of $n$, and there are $n/m$ dihedral subgroups $\langle a^{n/m}, b \rangle, \langle a^{n/m}, b^2 \rangle, \ldots, \langle a^{n/m}, a^{n/m-1} b \rangle$ containing $C_m$ as a subgroup of index 2. If $n/m$ is odd, then these are all conjugate in $D_n$. As a representative of their conjugacy class, we take $D_m = \langle a^{n/m}, b\rangle$. If $n/m$ is even, then there are two conjugacy classes. As representatives, we take $D_m = \langle a^{n/m}, b\rangle$ and $D'_m = \langle a^{n/m}, ab\rangle$.

Let $T_k = \mathbb{C}$, $1 \le k < n/2$, be the $D_n$-representation on which $D_n$ acts by $a \cdot z = \xi^k z$, $b \cdot z = \overline{z}$, $z \in S_k$, where $\xi = \exp(2\pi\sqrt{-1}/n)$. These $T_k$ are all (nonisomorphic) 2-dimensional irreducible representations over $\mathbb{R}$ [45]. It follows that Ker $T_k = C_{(k,n)}$. and

$$\text{Iso } T_k = \{D_n, \langle a^{n/(k,n)}, a^t b\rangle, \langle a^{n/(k,n)}\rangle \mid 0 \le t \le n-1\}.$$

Note also that

$$\dim T_k^H = \begin{cases} 2 \text{ if } H \le C_{(k,n)} \\ 1 \text{ if } H \text{ is conjugate to } D_{(k,n)} \text{ or } D'_{(k,n)} \\ 0 \text{ otherwise.} \end{cases}$$

*Proof of Theorem* 4.6. Let $k$ be an integer prime to $n$ with $1 < k < n/2$. Consider a pair $(T_1,\ T_k)$ of representations of $D_n$. It is easily seen that $(T_1,\ T_k)$ satisfies conditions $(C_{T_1,T_k})$ and $(I_{T_1,T_k})$. We show that there is no $D_n$-isovariant map from $T_1$ to $T_k$. Suppose that there is a $D_n$-isovariant map from $T_1$ to $T_k$ for some $k$; then, by normalization, one has a $D_n$-isovariant map $f : ST_1 \to ST_k$. Note that $ST_1^{>1} = ST_k^{>1} = \{\exp(\pi t\sqrt{-1}/n) \mid 0 \le t \le n-1\}$. Take $x = 1$ and $y = \exp(\pi\sqrt{-1}/n)$, then the isotropy subgroup at $x$ in $ST_1$ is $\langle b\rangle$, and also the isotropy subgroup at $y$ in $ST_1$ is $\langle ab\rangle$. Since $ST_k^{\langle b\rangle} = \{\pm 1\} \subset \mathbb{C}$, it follows that $f(1) = \pm 1$. Composing, if necessary, the antipodal map $z \mapsto -z$ on $ST_k$ with $f$, we may assume $f(1) = 1$. Let $A$ be the shorter arc joining $x$ with $y$ in $ST_1$. Since every point of the interior of $A$ has trivial isotropy subgroup, it follows that $f(A \setminus \{x, y\})$ is contained in $ST_k \setminus ST_k^{>1}$; hence $f(y)$ must be $y$ or $\overline{y}$. However the isotropy subgroup at $y$ (resp. $\overline{y}$) in $ST_k$ is equal to $\langle a^r b\rangle$ (resp. $\langle a^{-r} b\rangle$), where $r$ is a positive integer with $kr \equiv 1 \bmod n$, but it is not equal to $\langle ab\rangle$, since $k \not\equiv \pm 1 \bmod n$. This contradicts the isovariance of $f$. Thus the proof is complete. □

# 5 The existence of isovariant maps from a rational homology sphere with pseudofree $S^1$-action to a linear $S^1$-sphere

Let $G = S^1$ $(\subset \mathbb{C})$. Let $T_i$ $(= \mathbb{C})$ be the irreducible representation of $S^1$ defined by $g \cdot z = g^k z$. Let $M$ be a rational homology sphere with *pseudofree* $S^1$-action.

**Definition** (Montgomery-Yang [28]). An $S^1$-action on $M$ is *pseudofree* if

(1) the action is effective, and

(2) the singular set $M^{>1} := \bigcup_{1 \neq H \leq S^1} M^H$ is not empty and consists of finitely many exceptional orbits.

Here an orbit $G(x)$ is called exceptional if $G(x) \cong S^1/D$, $(1 \neq D < S^1)$ [6].

*Remark.* Other meanings for the term "pseudofree action" appear in the literature.

**Example 5.1.** Let $V = T_p \oplus T_q \oplus T_r$. Then the $S^1$-action on $SV$ is pseudofree. Indeed it is clearly effective, and

$$\begin{aligned} SV^{>1} &= ST_p \coprod ST_q \coprod ST_r \\ &\cong S^1/C_p \coprod S^1/C_q \coprod S^1/C_r. \end{aligned}$$

*Remark.* There are many "exotic" pseudofree $S^1$-actions on high-dimensional homotopy spheres [28], [42].

Then the following isovariant Borsuk-Ulam type result can be verified.

**Theorem 5.2** ([33]). *Let $M$ be a rational homology sphere with* pseudofree *$S^1$-action and $SW$ a linear $S^1$-sphere. There is an $S^1$-isovariant map $f : M \to SW$ if and only if*

(I): $\operatorname{Iso} M \subset \operatorname{Iso} SW$,

(PF1): $\dim M - 1 \leq \dim SW - \dim SW^H$ *when $H$ is a nontrivial subgroup which is contained in some $D \in \operatorname{Iso} M$,*

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(PF2): $\dim M + 1 \leq \dim SW - \dim SW^H$ *when* $H$ *is a nontrivial subgroup which is not contained in any* $D \in \operatorname{Iso} M$.

We give some examples. Let $p$, $q$, $r$ be pairwise coprime integers greater than 1.

**Example 5.3.** There is no $S^1$-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

*Proof.* Condition (PF1) is not fulfilled. □

*Remark.* There is an $S^1$-equivariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

**Example 5.4.** There is an $S^1$-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

*Proof.* One can see that $\operatorname{Iso} M = \{1, C_p, C_q, C_r\}$ and

$$\operatorname{Iso} SW = \{1, C_p, C_q, C_r, C_{pq}, C_{qr}, C_{rp}\}.$$

Hence it is easily seen that (PF1) and (PF2) are fulfilled and $\operatorname{Iso} M \subset \operatorname{Iso} SW$. By the theorem above, there is an $S^1$-isovariant map. □

From this, we obtain an isovariant map in the case of Example 4.5(4).

**Corollary 5.5.** *There is an* $C_{pqr}$*-isovariant map*

$$f : S(U_p \oplus U_q \oplus U_r) \to S(U_1 \oplus U_{pq} \oplus U_{qr} \oplus U_{rp}).$$

*Proof.* By restricting $f$ in Example 5.4 to the $C_{pqr}$-action, one has the desired map. □

## 5.1 Proof of Theorem 5.2 (outline)

We shall give an outline of the proof of Theorem 5.2. Full details can be found in [33]. Set $SW_{\text{free}} := SW \setminus SW^{>1}$. Note that $S^1$ acts freely on $SW_{\text{free}}$. Let $N_i$ be an $S^1$-tubular neighborhood of each exceptional orbit in $M$. By the slice theorem, $N_i$ is identified with $S^1 \times_{D_i} DU_i$ $(1 \le i \le r)$, where $D_i$ is the isotropy group of the exceptional orbit and $U_i$ is the slice $D_i$-representation. Set $X := M \setminus (\coprod_i \operatorname{int} N_i)$. Note that $S^1$ acts freely on $X$.

The "only if" part is proved by the (isovariant) Borsuk-Ulam theorem. Indeed for (PF1), take a point $x \in M$ with $G_x = D$ and a $D$-invariant closed neighborhood $B$ of $x$ which is $D$-diffeomorphic to some unit disk $DV$. Hence we obtain an $H$-isovariant map $f_{|SV} : SV \to SW$ by restriction. Applying the isovariant Borsuk-Ulam theorem to $f$, we obtain (PF1).

We next show (PF2). Since $f$ is isovariant, one sees that $f$ maps $M$ into $SW \setminus SW^H$. Since $SW \setminus SW^H$ is $S^1$-homotopy equivalent to $S(W^{H^\perp})$, one obtains an $S^1$-map $g : M \to S(W^{H^\perp})$. By Corollary 2.3, condition (PF2) follows.

To show the converse, we use the equivariant obstruction theory. We recall the following result.

**Lemma 5.6.** *There is an $S^1$-isovariant map $\tilde{f}_i : N_i \to SW$.*

*Proof.* Let $N_i = N \cong_G S^1 \times_D DV \subset M$, where $D$ is the isotropy group of the exceptional orbit and $V$ is the slice representation. Similarly take a closed $S^1$-tubular neighborhood $N'$ of an exceptional orbit with isotropy group $D$, and set $N' \cong_G S^1 \times_D DV' \subset SW$. By (PF1), one sees that $\dim SV + 1 \le \dim SV' - \dim SV'^{>1}$. Since $D$ acts freely on $SV$, there is a $D$-map $g : SV \to SV' \setminus SV^{>1} \subset SW$ by Corollary 2.8, which leads to a $D$-isovariant map $g : SV \to SW$. Taking a cone, we have a $D$-isovariant map $\tilde{g} : DV \to DV'$, and hence an $S^1$-isovariant map $\overline{f} = S^1 \times_D \tilde{g} : N \to N' \subset SW$. □

Set $f_i := \tilde{f}_i|_{\partial N_i} : \partial N_i \to SW_{\text{free}}$, and $f := \coprod_i f_i : \partial X \to SW_{\text{free}}$. If $f$ is extended to an $S^1$-map $F : X \to SW_{\text{free}}$, by gluing the maps, we obtain an $S^1$-isovariant map

$$F \cup (\coprod_i \tilde{f}_i) : M \to SW.$$

Thus we need to investigate the extendability of an $S^1$-map $f : \partial X \to SW_{\text{free}}$ to $F : X \to SW_{\text{free}}$. Equivariant obstruction theory [10] answers this question. A standard computation shows

**Lemma 5.7** ([33], [38]). *Set* $d = \dim SW - \dim SW^{>1}$.

(1) $SW_{\text{free}}$ *is* $(d-2)$*-connected and* $(d-1)$*-simple.*

(2) $\pi_{d-1}(SW_{\text{free}}) \cong H_{d-1}(SW_{\text{free}})) \cong \oplus_{H\in\mathcal{A}}\mathbb{Z}$, *where*

$$\mathcal{A} := \{H \in \operatorname{Iso} SW \mid \dim SW^H = \dim SW^{>1}\}$$

*and the generators are represented by* $S(W^{H^\perp})$, $H \in \mathcal{A}$.

By noticing that $\dim M - 1 \le d$ by (PF1) and (PF2), the obstruction $\mathfrak{o}_{S^1}(f)$ to the existence of an $S^1$-map $F : X \to SW_{\text{free}}$ lies in the equivariant cohomology group

$$\mathfrak{H}^d_{S^1}(X, \partial X; \pi_{d-1}(SW_{\text{free}})) \cong H^d(X/S^1, \partial X/S^1; \pi_{d-1}(SW_{\text{free}})).$$

If $\dim M - 1 < d$ (i.e., $\dim X/S^1 < d$), then one sees that

$$H^*(X/S^1, \partial X/S^1; \pi_{*-1}(SW_{\text{free}})) = 0$$

by dimensional reasons. Hence the obstruction vanishes and there exists an extension $F : X \to SW_{\text{free}}$.

We hereafter assume that $\dim M - 1 = d$ (i.e., $\dim X/S^1 = d$). The computation of the obstruction is executed by the multidegree.

**Definition.** Let $N = S^1 \times_D DU \subset M$, $1 \neq D \in \operatorname{Iso} M$. Assume that $\dim M - 1 = \dim U = d$. Let $f : \partial N \to SW_{\text{free}}$ be an $S^1$-map, and consider the $D$-map $\overline{f} = f|_{SU} : SU \to SW_{\text{free}}$. Then the multidegree of $f$ is defined by

$$\operatorname{mDeg} f := \overline{f}_*([SU]) \in \oplus_{H\in\mathcal{A}}\mathbb{Z},$$

under the natural identification $H_{d-1}(SW_{\text{free}})$ with $\oplus_{H\in\mathcal{A}}\mathbb{Z}$.

The obstruction $\mathfrak{o}_{S^1}(f)$ is described by the multidegree as follows.

**Proposition 5.8** ([33]). *Let $F_0 : X \to SW_{\text{free}}$ be a fixed $S^1$-map; this map always exists, however, it is not necessary to extend it to an isovariant map on $M$. Set $f_{0,i} = F_0|_{\partial N_i}$. Then*

$$\mathfrak{o}_{S^1}(f) = \sum_{i=1}^{r} (\text{mDeg}\, f_i - \text{mDeg}\, f_{0,i})/|D_i|,$$

*under the natural identification $H_{d-1}(SW_{\text{free}})$ with $\oplus_{H \in \mathcal{A}} \mathbb{Z}$.*

*Remark.* It follows from the equivariant Hopf type result [33] that

$$\text{mDeg}\, f_i - \text{mDeg}\, f_{0,i} \in \oplus_{H \in \mathcal{A}} |D_i| \mathbb{Z}.$$

In addition, the following extendability result is known.

**Proposition 5.9** ([33]). *Let $N = S^1 \times_D DV$ be as before and $f : \partial N \to SW_{\text{free}}$ be an $S^1$-map. Set* $\text{mDeg}\, f = (d_H(f))$.

(1) *$f : \partial N \to SW_{\text{free}}$ is extendable to an $S^1$-isovariant map $\tilde{f} : N \to SW$ if and only if $d_H(f) = 0$ for any $H \in \mathcal{A}$ with $H \not\leq D$.*

(2) *For any extendable $f$ and for any $(a_H) \in \oplus_{H \in \mathcal{A}} |D| \mathbb{Z}$ satisfying $a_H = 0$ for $H \in \mathcal{A}$ with $H \not\leq D$, there exists an $S^1$-map $f' : \partial N \to SW_{\text{free}}$ such that $f'$ is extendable to an $S^1$-isovariant map $\tilde{f}' : N \to SW$ and* $\text{mDeg}\, f' = \text{mDeg}\, f + (a_H)$.

Using these propositions, one can see that there are $S^1$-isovariant maps $f_i : \partial N_i \to SW$ such that $\coprod_i f_i$ extends both on $X$ and on $\coprod_i N_i$ as isovariant maps. Thus an isovariant map from $M$ to $SW$ is constructed.

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