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# THE SMITH HOMOLOGY AND BORSUK-ULAM TYPE THEOREMS

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**Abstract**

Let $k$ be a positive integer greater than 1 and $C_k$ be the cyclic group of order $k$. Let $X$ be an arcwise connected free $C_k$-space and $Y$ be a Hausdorff free $C_k$-space. If there exists a positive integer $n$ such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \leq q \leq n$ and $H_{n+1}(Y/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then there is no continuous $C_k$-map from $X$ to $Y$.

We also prove a definable version of this topological version in an o-minimal expansion of $\mathcal{N} = (R, +, \cdot, <, ...)$ of a real closed field $R$.

## 1. Introduction

Let $k$ be a positive integer greater than 1 and $C_k$ be the cyclic group of order $k$. Let $\mathbb{S}^n$ be the $n$-dimensional unit sphere of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with the antipodal $C_2$-action. From the viewpoint of transformation groups, the classical Borsuk-Ulam theorem states that if there exists a continuous $C_2$-map from $\mathbb{S}^n$ to $\mathbb{S}^m$, then $n \leq m$. There are several equivalent statements of it and many related generalizations (e.g., [2], [13], [14], [15], [17]).

The classical Borsuk-Ulam theorem is generalized to topological spaces by several authors. For example, Walker [21], Pergher et al. [18]. They prove non-existence of continuous $C_2$-maps between free $C_2$-spaces under certain homological conditions on the free $C_2$-spaces. Essentially they use the Smith-Gysin exact sequence in their proof. If $k$ is a positive integer greater than 1, then several $C_k$-versions of the classical Borsuk-Ulam theorem are discussed in Kobayashi [11] and Hemmi et al. [7].

In this paper, we use the Smith homology (c.f. [10]) which is a useful simple tool to study $C_k$-versions of the classical Borsuk-Ulam theorem in the topological setting and the definable setting. The Smith exact sequence which is expressed by using the Smith homology is a generalization of the Smith-Gysin exact sequence. By using this, we can give a simple proof of a $C_k$-version of the classical Borsuk-Ulam theorem. In this paper, we prove the following generalized Borsuk-Ulam theorem which is a generalization of [21], [18], [11] and [7].

**Theorem 1.1.** *Let $X$ be an arcwise connected free $C_k$-space and $Y$ be a Hausdorff free $C_k$-space. If there exists a positive integer $n$ such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \le q \le n$ and $H_{n+1}(Y/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then there is no continuous $C_k$-map from $X$ to $Y$. Here this homology means the singular homology.*

The following remark shows that we cannot take $k = 1$ and $k = \infty$ in Theorem 1.1.

**Remark 1.2.** (1) Let $n \in \mathbb{N}$ and $Y$ be a one-point set. Then the constant map from $\mathbb{R}^n$ to $Y$ is a continuous map and $\mathbb{R}^n$ and $Y$ satisfy the conditions on Theorem 1.1.

(2) Let $n \in \mathbb{N}$. Then $\mathbb{R}^n$ has the free $\mathbb{Z}$-action defined by $\mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$, $(g, x_1, ..., x_n) \mapsto (g + x_1, x_2, ..., x_n)$. Therefore, $\mathbb{R}^n$ and $\mathbb{R}$ satisfy the assumptions on Theorem 1.1 and the map $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x_1, ..., x_n) = x_1$ is a continuous $\mathbb{Z}$-map.

Let $k$ be a prime. For a topological space $Y$, let $D = \{(y_1, ..., y_k) \in Y \times \cdots \times Y \mid y_1 = \cdots = y_k\}$ be the diagonal and write $Y^* = Y \times \cdots \times Y - D$ admitting the free $C_k$-action defined by $g(y_1, y_2, ..., y_k) = (y_2, y_3, ..., y_k, y_1)$, where $g$ generates $C_k$.

**Theorem 1.3.** *Let $k$ be a prime and $X$ be an arcwise connected free $C_k$-space. If there exists a positive integer $n$ such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \le q \le n$ and $Y$ is a Hausdorff space with $H_{n+1}(Y^*/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then every continuous map $f : X \to Y$ has a $C_k$-coincidence point, that is, a point $x$ such that $f(x) = f(gx) = \cdots = f(g^{k-1}x)$, where $g$ is a generator of $C_k$.*

We can consider the definable versions of Theorem 1.1 and Theorem 1.3 in an o-minimal expansion $\mathcal{N} = (R, +, \cdot, <, ...)$ of a real closed field $R$.

Many results in the semialgebraic geometry hold in the o-minimal setting and there exist uncountably many o-minimal expansions of the standard structure of the field $\mathbb{R}$ of real numbers ([19]). See also [4], [6], [12] for examples and constructions of o-minimal structures. General references on them are [3], [5], [20]. In this paper, "definable" means "definable with parameters in $\mathcal{N}$", every definable object is considered in $\mathcal{N}$ and each definable map is continuous unless otherwise stated.

Let $S^n$ denote the $n$-dimensional unit sphere of $R^{n+1}$. If $R = \mathbb{R}_{alg}$, then $S^n$ is neither arcwise connected nor connected. Thus we cannot apply [21], [18], [11], [7] and Theorem 1.1 even if $X = S^{2n+1}$ and $Y = S^{2m+1}$.

The singular definable homology is introduced in [22]. Using the singular definable homology, we have the following theorem which is a definable version of Theorem 1.1.

**Theorem 1.4.** *Let X be a definably connected definable set with a free definable $C_k$-action. If there exists a positive integer n such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \le q \le n$ and Y is a definable set with a free definable $C_k$-action such that $H_{n+1}(Y/C_k ; \mathbb{Z}/k\mathbb{Z}) = 0$, then there is no definable $C_k$-map from X to Y. Here this homology means the singular definable homology.*

Note that a definably connected definable set is not necessarily connected and a definable set is definably connected if and only if definably arcwise connected. Here a definable set $X$ is definably arcwise connected if for every two points $x, y \in X$, there exists a definable map $f$ from the closed unit interval $[0, 1]_R$ of $R$ to $X$ such that $x = f(0)$ and $y = f(1)$.

In the definable setting, we have the following simple sufficient condition on $Y$ which implies $H_{n+1}(Y/C_k ; \mathbb{Z}/k\mathbb{Z}) = 0$.

If $Y$ is a definable set with a definable $C_k$-action, then by Corollary 10.2.18 in [3], $Y/C_k$ is a definable set and the orbit map $\pi : Y \to Y/C_k$ is definable. If dim $Y \le n$, then by Corollary 4.1.6 in [3], $\dim Y/C_r \le n$. Thus if $\dim Y \le n$, then $H_{n+1}(Y/C_k ; \mathbb{Z}/k\mathbb{Z}) = 0$.

**Corollary 1.5.** (1) *Suppose that $k \ge 3$ and that $C_k$ acts on $S^{2m+1}$ and $S^{2n+1}$ definably and freely. If there exists a definable $C_k$-map $f : S^{2m+1} \to S^{2n+1}$, then $m \le n$.*

(2) *If $S^m$ and $S^n$ have free definable $C_2$-actions and there exists a definable $C_2$-map $f : S^m \to S^n$, then $m \le n$.*

Corollary 1.5 is a generalization of Theorem 1.1 [16].

Using Theorem 1.4, we have the following theorem.

**Theorem 1.6.** *Let $k$ be a prime and $X$ be a definably connected definable set with a free definable $C_k$-action. Assume that there exists a positive integer $n$ such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \leq q \leq n$. If $Y$ is a definable set with $H_{n+1}(Y^*/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then every definable map $f : X \to Y$ has a $C_k$-coincidence point, that is, a point $x$ such that $f(x) = f(gx) = \cdots = f(g^{k-1}x)$, where $g$ is a generator of $C_k$.*

## 2. Proof of Theorem 1.1 and Theorem 1.3

We first prove Theorem 1.1.

Let $\mathbb{Z}/k\mathbb{Z}[C_k]$ denote the group ring of $C_k$ over $\mathbb{Z}/k\mathbb{Z}$. For any $q \in \mathbb{N} \cup \{0\}$, the $q$-dimensional chain group $C_q(X; \mathbb{Z}/k\mathbb{Z})$ has the standard $C_k$-action. Then this action induces $\mathbb{Z}/k\mathbb{Z}[C_k]$-action on $C_q(X; \mathbb{Z}/k\mathbb{Z})$.

Let $g$ be a generator of $C_k$, $\alpha = 1 + g + \cdots + g^{k-1}$, and $\beta = 1 - g$. Then by definition $\alpha\beta = \beta\alpha = 0$, for every $q$, $\alpha C_q(X; \mathbb{Z}/k\mathbb{Z})$ and $\beta C_q(X; \mathbb{Z}/k\mathbb{Z})$ are $\mathbb{Z}/k\mathbb{Z}[C_k]$-submodules of $C_q(X; \mathbb{Z}/k\mathbb{Z})$ and $\alpha\partial = \partial\alpha$, $\beta\partial = \partial\beta$, where $\partial$ is the boundary operator of $\{C_q(X; \mathbb{Z}/k\mathbb{Z})\}$. Therefore, $\{\alpha C_q(X; \mathbb{Z}/k\mathbb{Z})\}$ and $\{\beta C_q(X; \mathbb{Z}/k\mathbb{Z})\}$ are subchain complexes of $\{C_q(X; \mathbb{Z}/k\mathbb{Z})\}$.

**Proposition 2.1.** *For every $q$, the following two sequences are exact:*

$$0 \to \alpha C_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{i} C_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\beta} \beta C_q(X; \mathbb{Z}/k\mathbb{Z}) \to 0,$$

$$0 \to \beta C_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{j} C_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\alpha} \alpha C_q(X; \mathbb{Z}/k\mathbb{Z}) \to 0,$$

*where $i, j$ denote the inclusions and $\alpha$ (resp. $\beta$) stands for the multiplication of $\alpha$ (resp. $\beta$).*

**Proof.** Since $\beta \circ i = 0$, $\alpha \circ j = 0$, $\operatorname{Im} i \subset \operatorname{Ker} \beta$, $\operatorname{Im} j \subset \operatorname{Ker} \alpha$.

Let $s = \sum_j \sum_{i=0}^{k-1} n_{ji} g^i \sigma_j \in \operatorname{Ker} \beta$, where $g$ is a generator of $C_k$. If $l \neq l'$

and $0 \le i \le k-1$, then $g^i\sigma_l \ne \sigma_{l'}$. Since $\beta s = 0$, for any $j$, $\sum_{i=0}^{k-1} n_{ji} g^i(1-g)\sigma_j = 0$. Thus for every $j$, $\sum_{i=1}^{k-1}(n_{ji} - n_{j(i-1)})g^i\sigma_i + (n_{j0} - n_{j(k-1)})\sigma_j = 0$. Hence for each $j$, $n_{j0} = n_{j1} = \cdots = n_{jk-1}$. We set $n_j = n_{j0} (= n_{j1} = \cdots = n_{jk-1})$. Then we have $s = \sum_j n_j(1 + g + \cdots + g^{k-1})\sigma_j = \alpha \sum_j n_j \sigma_j \in \operatorname{Im} i$. Therefore, $\operatorname{Ker}\beta = \operatorname{Im} i$.

Let $s = \sum_j \sum_{i=0}^{k-1} n_{ji} g^i \sigma_j \in \operatorname{Ker}\alpha$. Since $\alpha s = \sum_j (n_{j0} + \cdots + n_{j(k-1)})(1 + \cdots + g^{k-1})\sigma_j = 0$, $n_{j0} + \cdots + n_{j(k-1)} = 0$.

Thus $s = \sum_j (n_{j0}(1-g) + (n_{j0} + n_{j1})g(1-g) + (n_{j0} + n_{j1} + n_{j2})g^2(1-g) + \cdots + (n_{j0} + n_{j1} + \cdots + n_{j(k-2)})g^{k-2}(1-g))\sigma_j \in \operatorname{Im} j$. Therefore, $\operatorname{Ker}\alpha = \operatorname{Im} j$. □

Let $H_q^{\alpha}(X, \mathbb{Z}/k\mathbb{Z})$ (resp. $H_q^{\beta}(X; \mathbb{Z}/k\mathbb{Z})$) denote the homology group induced from the chain complex $\{\alpha C_q(X; \mathbb{Z}/k\mathbb{Z})\}$ (resp. $\{\beta C_q(X; \mathbb{Z}/k\mathbb{Z})\}$). We call these homology groups the *Smith homology groups*.

By Proposition 2.1, we have the following theorem.

**Theorem 2.2.** *The following two sequences are exact:*

$$\cdots \to H_q^{\alpha}(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{i_*} H_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\beta_*} H_q^{\beta}(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\partial_*} H_{q-1}^{\alpha}(X; \mathbb{Z}/k\mathbb{Z}) \to \cdots$$

$$\cdots \to H_q^{\beta}(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{j_*} H_q(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\alpha_*} H_q^{\alpha}(X; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\partial'_*} H_{q-1}^{\beta}(X; \mathbb{Z}/k\mathbb{Z}) \to \cdots.$$

*In particular, if $k = 2$, then $\alpha = \beta$ and*

$$\cdots \to H_q^{\alpha}(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i_*} H_q(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\alpha_*} H_q^{\alpha}(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial_*} H_{q-1}^{\alpha}(X; \mathbb{Z}/2\mathbb{Z}) \to \cdots$$

*is exact.*

Let $p : E \to X$ be a continuous map. We say that $p$ has the *homotopy lifting property* if for any compact space $Z$, each homotopy $h : Z \times [0, 1] \to X$ and a continuous map $F : Z \to E$ such that $p \circ F(z) = h(z, 0)$, for all $z \in Z$, there exists a homotopy $H : Z \times [0, 1] \to E$ such that $p \circ H = h$ and $H(z, 0) = F(z)$, for all $z \in Z$, where $[0, 1]$ denotes the closed unit interval of $\mathbb{R}$.

**Proposition 2.3.** *Let $X$ be an arcwise connected Hausdorff free $C_k$-space. Then the orbit map $\pi : X \to X/C_k$ has the homotopy lifting property.*

**Proposition 2.4.** *If $Y$ is an arcwise connected Hausdorff free $C_k$-space, then for every $q$, $H_q^{\alpha}(Y, \mathbb{Z}/k\mathbb{Z}) \cong H_q(Y/C_k, \mathbb{Z}/k\mathbb{Z})$.*

**Proof.** We first show that the map $\alpha : C(Y; \mathbb{Z}/k\mathbb{Z}) \to C(Y; \mathbb{Z}/k\mathbb{Z})$ and the map $\pi_* : C(Y; \mathbb{Z}/k\mathbb{Z}) \to C(Y/C_k; \mathbb{Z}/k\mathbb{Z})$ induced from the orbit map $\pi : Y \to Y/C_k$ have the same kernel. Let $\sigma$ be a singular $s$-simplex of $Y$. We need only to consider elements of $C(C_k\sigma)$, since $C(Y) \cong \oplus_{[\sigma]\in\Delta(s)/C_k} C(C_k\sigma)$, where $\Delta(s)$ is the set of singular $s$-simplexes of $Y$ and $\Delta(s)/C_k$ is its orbit set under the induced action.

Since $\alpha\left(\sum n_i g^i \sigma\right) = \left(\sum n_i\right)\alpha(\sigma)$, $\alpha\left(\sum n_i g^i \sigma\right) = 0$ if and only if $\sum n_i = 0$, and similarly $\pi_*\left(\sum n_i g^i \sigma\right) = \left(\sum n_i\right)\pi \circ \sigma = 0$ if and only if $\sum n_i = 0$; therefore, both kernels coincide.

We next show that $\pi_*$ is surjective; namely, there is a lift $\tilde{\tau} : \Delta^s \to Y$ of $\tau : \Delta^s \to Y/C_k$, where $\Delta^s$ denotes the affine span of $(s+1)$-points which are affine independent. Since $\Delta^s$ is contractible, there is a homotopy $H' : \Delta^s \times [0, 1] \to \Delta^s$ such that $H'(-, 0) = c_{e_0}$ and $H'(-, 1) = id_{\Delta^s}$, where $c_{e_0}$ denotes the constant map whose value is $e_0 \in \Delta^s$. Then the composition $H = \tau \circ H'$ is a homotopy from the constant map $c_{\tau(e_0)}$ to $\tau$. Let $y_0$ be a point of $Y$ such that $\pi(y_0) = \tau(e_0)$, and $c_{y_0} : \Delta^s \to Y$ the constant map whose value is $y_0$. Since $H(-, 0) = \pi \circ c_{y_0}$, it follows from Proposition 2.3 that there exists a lift $\tilde{H} : \Delta^s \times [0, 1] \to Y$ of $H$ such that $\tilde{H}(-, 0) = c_{y_0}$. Then $\tilde{\tau} := \tilde{H}(-, 1)$ is a lift of $\tau = H(-, 1)$.

Since $\pi_*$ is surjective, $\alpha C(Y; \mathbb{Z}/k\mathbb{Z})$ and $C(Y/C_p; \mathbb{Z}/k\mathbb{Z})$ are isomorphic as chain complexes. Accordingly their homology groups are also isomorphic. □

**Proof of Theorem 1.1.** Assume that there exists a continuous $C_k$-map $f : X$

$\to Y$ under the conditions of Theorem 1.1. Since $X$ is arcwise connected, $f(X)$ is acrwise connected. Hence $f(X)$ is contained in an arcwise connected component of $Y$. Therefore it is sufficient to prove the case where $Y$ is arcwise connected.

We first prove the case where $k = 2$. Since $f$ is a continuous $C_2$-map, $\alpha f_{\#} = f_{\#}\alpha$.

For simplicity, we abbreviate the coefficient $\mathbb{Z}/2\mathbb{Z}$ in the singular homology. By Theorem 2.2, we have a commutative diagram

$$\begin{array}{ccccccccccc}
\to & H^{\alpha}_{n+1}(X) & \stackrel{\partial^X_*}{\to} & H^{\alpha}_{n}(X) & \stackrel{i^X_*}{\to} & H_n(X) & \stackrel{\alpha^X_*}{\to} & H^{\alpha}_{n}(X) & \stackrel{\partial^X_*}{\to} & H^{\alpha}_{n-1}(X) & \to \cdots \\
 & f^{\alpha}_* \downarrow & & f^{\alpha}_* \downarrow & & f_* \downarrow & & f^{\alpha}_* \downarrow & & f^{\alpha}_* \downarrow & \\
\to & H^{\alpha}_{n+1}(Y) & \stackrel{\partial^Y_*}{\to} & H^{\alpha}_{n}(Y) & \stackrel{i^Y_*}{\to} & H_n(Y) & \stackrel{\alpha^Y_*}{\to} & H^{\beta}_{n}(Y) & \stackrel{\partial^Y_*}{\to} & H^{\alpha}_{n-1}(Y) & \to \cdots
\end{array}$$

$$\begin{array}{cccccccccccccc}
\to & H^{\alpha}_{1}(X) & \stackrel{i^X_*}{\to} & H_1(X) & \stackrel{\alpha^X_*}{\to} & H^{\alpha}_{1}(X) & \stackrel{\partial^X_*}{\to} & H^{\alpha}_{0}(X) & \stackrel{i^X_*}{\to} & H_0(X) & \stackrel{\alpha^X_*}{\to} & H^{\alpha}_{0}(X) & \to & 0 \\
 & f^{\alpha}_* \downarrow & & f_* \downarrow & & f^{\alpha}_* \downarrow & & f^{\alpha}_* \downarrow & & f_* \downarrow & & f^{\alpha}_* \downarrow & & f_* \downarrow \\
\to & H^{\alpha}_{1}(Y) & \stackrel{i^Y_*}{\to} & H_1(Y) & \stackrel{\alpha^Y_*}{\to} & H^{\alpha}_{1}(Y) & \stackrel{\partial^Y_*}{\to} & H^{\alpha}_{0}(Y) & \stackrel{i^Y_*}{\to} & H_0(Y) & \stackrel{\alpha^Y_*}{\to} & H^{\alpha}_{0}(Y) & \to & 0
\end{array}$$

with exact rows.

By definition, $(i^X_*)_0 = 0$ and $(i^Y_*)_0 = 0$. Thus $(\alpha^X_*)_0 : H_0(X) \to H^{\alpha}_0(X)$ and $(\alpha^Y_*)_0 : H_0(Y) \to H^{\alpha}_0(Y)$ are isomorphisms. By assumption, $H_0(X) \cong \mathbb{Z}/2\mathbb{Z}$. Hence $H_0(X) \cong H^{\alpha}_0(X) \cong \mathbb{Z}/2\mathbb{Z}$. Similarly, $H_0(Y) \cong H^{\alpha}_0(Y) \cong \mathbb{Z}/2\mathbb{Z}$. Since $(f_*)_0 : H_0(X) \to H_0(Y)$ is an isomorphism and $(\alpha^X_*)_0 \circ (f_*)_0 = (f^{\alpha}_*)_0 \circ (\alpha^X_*)_0$, $(f^{\alpha}_*)_0 : H^{\alpha}_0(X) \to H^{\alpha}_0(Y)$ is an isomorphism. Since $(i^X_*)_0 = 0$, we have $\mathrm{Im}(\partial^X_*)_1 = \mathrm{Ker}\,(i^X_*)_0 = H^{\alpha}_0(X)$. Thus we see that $(\partial^Y_*)_1 \circ (f^{\alpha}_*)_1 = (f^{\alpha}_*)_0 \circ (\partial^X_*)_1 :$ $H^{\alpha}_1(X) \to H^{\alpha}_0(Y)$ is a non-zero homomorphism. Hence $(f^{\alpha}_*)_1 : H^{\alpha}_1(X) \to H^{\alpha}_1(Y)$ is a non-zero homomorphism. Using the assumptions on $X$, we see that $(\partial^X_*)_q :$ $H^{\alpha}_q(X) \to H^{\alpha}_{q-1}(X)$ is an isomorphism for each $1 \le q \le n$. Using this fact and by induction, we can prove that $(f^{\alpha}_*)_q : H^{\alpha}_q(X) \to H^{\alpha}_q(Y)$ is a non-zero homomorphism for each $0 \le q \le n$.

By Proposition 2.4, $H_{n+1}^{\alpha}(Y) \cong H_{n+1}(Y/C_p)$. Thus $H_{n+1}^{\alpha}(Y) = 0$. Hence $(i_*^Y)_n : H_n^{\alpha}(Y) \to H_n(Y)$ is injective and $(i_*^X)_n \circ (f_*^{\alpha})_n : H_n^{\alpha}(X) \to H_n(Y)$ is a non-zero homomorphism.

On the other hand, since $H_n(X) = 0$, $(i_*^Y)_n \circ (f_*^{\alpha})_n = (f_*)_n \circ (i_*^X)_n = 0$. This contradiction proves the theorem in this case.

Next, we prove the case where $k > 2$. For simplicity, we abbreviate the coefficient $\mathbb{Z}/k\mathbb{Z}$ in the singular homology. By Theorem 2.2, we have two commutative diagrams

$$\begin{array}{ccccccccccc}
\to & H_n^{\alpha}(X) & \xrightarrow{i_*^X} & H_n(X) & \xrightarrow{\beta_*^X} & H_n^{\beta}(X) & \xrightarrow{\partial_*^X} & H_{n-1}^{\alpha}(X) & \to & \cdots \\
 & f_*^{\alpha} \downarrow & & f_* \downarrow & & f_*^{\beta} \downarrow & & f_*^{\alpha} \downarrow & & \\
\to & H_n^{\alpha}(Y) & \xrightarrow{i_*^Y} & H_n(Y) & \xrightarrow{\beta_*^Y} & H_n^{\beta}(Y) & \xrightarrow{\partial_*^Y} & H_{n-1}^{\alpha}(Y) & \to & \cdots
\end{array}$$

$$\begin{array}{ccccccccccccccc}
\to & H_1^{\alpha}(X) & \xrightarrow{i_*^X} & H_1(X) & \xrightarrow{\beta_*^X} & H_1^{\beta}(X) & \xrightarrow{\partial_*^X} & H_0^{\alpha}(X) & \xrightarrow{i_*^X} & H_0(X) & \xrightarrow{\beta_*^X} & H_0^{\beta}(X) & \to & 0 \\
 & f_*^{\alpha} \downarrow & & f_* \downarrow & & f_*^{\beta} \downarrow & & f_*^{\alpha} \downarrow & & f_* \downarrow & & f_*^{\beta} \downarrow & & f_* \downarrow \\
\to & H_1^{\alpha}(Y) & \xrightarrow{i_*^Y} & H_1(Y) & \xrightarrow{\beta_*^Y} & H_1^{\beta}(Y) & \xrightarrow{\partial_*^Y} & H_0^{\alpha}(Y) & \xrightarrow{i_*^Y} & H_0(Y) & \xrightarrow{\beta_*^Y} & H_0^{\beta}(Y) & \to & 0
\end{array}$$

and

$$\begin{array}{ccccccccccccc}
\to & H_{n+1}^{\alpha}(X) & \xrightarrow{\partial_*'^X} & H_n^{\beta}(X) & \xrightarrow{j_*^X} & H_n(X) & \xrightarrow{\alpha_*^X} & H_n^{\alpha}(X) & \xrightarrow{\partial_*'^X} & H_{n-1}^{\beta}(X) & \to & \cdots \\
 & f_*^{\alpha} \downarrow & & f_*^{\beta} \downarrow & & f_* \downarrow & & f_*^{\alpha} \downarrow & & f_*^{\beta} \downarrow & & \\
\to & H_{n+1}^{\alpha}(Y) & \xrightarrow{\partial_*'^Y} & H_n^{\beta}(Y) & \xrightarrow{j_*^Y} & H_n(Y) & \xrightarrow{\alpha_*^Y} & H_n^{\alpha}(Y) & \xrightarrow{\partial_*'^Y} & H_{n-1}^{\beta}(Y) & \to & \cdots
\end{array}$$

$$\begin{array}{ccccccccccccccc}
\to & H_1^{\beta}(X) & \xrightarrow{j_*^X} & H_1(X) & \xrightarrow{\alpha_*^X} & H_1^{\alpha}(X) & \xrightarrow{\partial_*'^X} & H_0^{\beta}(X) & \xrightarrow{j_*^X} & H_0(X) & \xrightarrow{\alpha_*^X} & H_0^{\alpha}(X) & \to & 0 \\
 & f_*^{\beta} \downarrow & & f_* \downarrow & & f_*^{\alpha} \downarrow & & f_*^{\beta} \downarrow & & f_* \downarrow & & f_*^{\alpha} \downarrow & & f_* \downarrow \\
\to & H_1^{\beta}(Y) & \xrightarrow{j_*^Y} & H_1(Y) & \xrightarrow{\alpha_*^Y} & H_1^{\alpha}(Y) & \xrightarrow{\partial_*'^Y} & H_0^{\beta}(Y) & \xrightarrow{j_*^Y} & H_0(Y) & \xrightarrow{\alpha_*^Y} & H_0^{\alpha}(Y) & \to & 0
\end{array}$$

with exact rows.

We easily see that $(i_*^X)_0 = 0$ and $(i_*^Y)_0 = 0$. Thus $(\beta_*^X)_0 : H_0(X) \to H_0^{\beta}(X)$ and $(\beta_*^Y)_0 : H_0(Y) \to H_0^{\beta}(Y)$ are isomorphisms. Since $(f_*)_0 : H_0(X) \to H_0(Y)$ is an isomorphism, $(f_*^{\beta})_0 : H_0^{\beta}(X) \to H_0^{\beta}(Y)$ is an isomorphism. Similarly, we see that $(f_*^{\alpha})_0 : H_0^{\alpha}(X) \to H_0^{\alpha}(Y)$ is an isomorphism from the second diagram. Since $H_1(X) = 0$ and $(i_*^X)_0 = 0$, $(\partial_*^X)_1 : H_1^{\beta}(X) \to H_0^{\alpha}(X)$ is an isomorphism.

Similarly $(\partial'^X_*)_1 : H^\alpha_1(X) \to H^\beta_0(X)$ is an isomorphism. Since $(\partial^Y_*)_1 \circ (f^\beta_*)_1 = (f^\alpha_*)_0 \circ (\partial^X_*)_1$ and $(\partial'^Y_*)_1 \circ (f^\alpha_*)_1 = (f^\beta_*)_0 \circ (\partial'^X_*)_1$, $(f^\alpha_*)_1 : H^\alpha_1(X) \to H^\alpha_1(Y)$ and $(f^\beta_*)_1 : H^\alpha_1(X) \to H^\beta_1(Y)$ are non-zero homomorphisms. By induction, we can prove that $(f^\alpha_*)_q : H^\alpha_q(X) \to H^\alpha_q(Y)$ and $(f^\beta_*)_q : H^\alpha_1(X) \to H^\beta_q(Y)$ are non-zero homomorphism for each $0 \le q \le n$. By Proposition 2.4, $H^\alpha_{n+1}(Y) \cong H_{n+1}(Y/C_p)$. Hence $H^\alpha_{n+1}(Y/C_p) = 0$ and $(j^Y_*)_n : H^\beta_n(Y) \to H_n(Y)$ is injective. Therefore, $(j^Y_*)_n \circ (f^\beta_*)_n$ is a non-zero homomorphism.

On the other hand, $(j^Y_*)_n \circ (f^\beta_*)_n = (f_*)_n \circ (j^X_*)_n = 0$ because $H_n(X) = 0$. This is a contradiction. Therefore, the proof is complete. □

**Proof of Theorem 1.3.** Suppose that there is no point $x \in X$ such that $f(x) = f(gx) = \cdots = f(g^{k-1}x)$. Then the map $F : X \to Y^*$ defined by $F(x) = (f(x), f(gx), \ldots, f(g^{k-1}x))$ is a continuous $C_k$-map. This contradicts Theorem 1.1. □

### 3. Proof of Theorem 1.4 and Theorem 1.6

To prove Theorem 1.4, we need a definable version of Proposition 2.4.

Definable fiber bundles are introduced in [9].

**Proposition 3.1** (Corollary 1.5 [8]). *Let $X$ be a definable set with a free definable $C_k$-action. Then $(X, \pi, X/C_k, C_k)$ is a principal definable $C_k$-fiber bundle, where $\pi : X \to X/C_k$ denotes the orbit map. In particular, $\pi : X \to X/C_k$ is a definable covering map.*

Let $p : E \to X$ be a definable map. We say that $p$ has the *definable homotopy lifting property* if for any definable set $Y$, each definable homotopy $h : Y \times [0, 1]_R \to X$ and a definable map $F : Y \to E$ such that $p \circ F(y) = h(y, 0)$ for all $y \in Y$, there exists a definable homotopy $H : Y \times [0, 1]_R \to E$ such that $p \circ H = h$ and $H(y, 0) = F(y)$ for all $y \in Y$.

**Theorem 3.2** (Proposition 4.10 [1]). *Every definable covering map has the definable homotopy lifting property.*

The following corollary is a definable version of Proposition 2.4.

**Corollary 3.3.** *Let $X$ be a definable set with a free definable $C_k$-action. Then the orbit map $\pi : X \to X/C_k$ has the definable homotopy lifting property.*

**Proof of Theorem 1.4 and Theorem 1.6.** Using Corollary 3.3 instead of Proposition 2.4, we can prove Theorem 1.4 by a way similar to the proof of Theorem 1.1. A similar proof of Theorem 1.3 proves Theorem 1.6. □

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# A survey of Borsuk-Ulam type theorems for isovariant maps

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ABSTRACT. In this article, we shall survey isovariant Borsuk-Ulam type theorems, which are interpreted as nonexistence results on isovariant maps from the viewpoint of equivariant topology or transformation group theory. We also discuss the existence problem of isovariant maps between representations as the converse of the isovariant Borsuk-Ulam theorem.

## 1 Introduction – backgrounds

Ever since K. Borsuk [5] proved the celebrated antipodal theorem, called the Borsuk-Ulam theorem, this theorem has attracted many researchers and has been generalized as Borsuk-Ulam type theorems, because it is not only beautiful but also has many interesting applications in several fields of mathematics like topology, nonlinear analysis and combinatorics. The original Borsuk-Ulam theorem is stated as follows:

**Proposition 1.1.** *For any continuous map $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$.*

Let $C_2$ be a cyclic group of order 2. Consider $C_2$-spheres $S^m$ and $S^n$ on which $C_2$ acts antipodally. By means of equivariant topology, the Borsuk-Ulam theorem is restated as follows:

**Proposition 1.2.** *If there is a $C_2$-map $f : S^m \to S^n$, then $m \leq n$. In other words, if $m > n$, then there is no $C_2$-map from $S^m$ to $S^n$.*

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Thus, in the context of equivariant topology, Borsuk-Ulam type theorems are thought of as nonexistence results on $G$-maps. Such results are implicitly and explicitly applied in several mathematical problems; for example, a Borsuk-Ulam type result plays an important role in the proof of Furuta's 10/8-theorem [14] in 4-dimensional topology. Lovász [21] succeeded in proving Kneser's conjecture in combinatorics using the Borsuk-Ulam theorem. Matoušek [27] also illustrates several applications to combinatorics. Clapp [9] applied a Borsuk-Ulam type theorem to a certain problem in nonlinear analysis. Further results and applications on the Borsuk-Ulam theorem can be found in excellent survey articles [46], [47].

Wasserman [49] first considered an isovariant version of the Borsuk-Ulam theorem. Nagasaki [31], [33], [34] and Nagasaki-Ushitaki [38] also studied isovariant Borsuk-Ulam type theorems. For example, Wasserman's results imply the following.

**Proposition 1.3.** *Let $G$ be a solvable compact Lie group and $V$, $W$ real $G$-representations. If there exists a $G$-isovariant map from $V$ to $W$, then the inequality*

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

*holds. Here $V^G$ denotes the $G$-fixed point set of $V$.*

In this article, we shall survey isovariant Borsuk-Ulam type theorems and related topics, in particular, we shall discuss the existence or nonexistence problem for isovariant maps between $G$-representations or more general $G$-spaces. This article is organized as follows. In Section 2, after recalling some *equivariant* Borsuk-Ulam type theorems, we shall show the isovariant Borsuk-Ulam theorem for semilinear actions which is a generalization of Proposition 1.3. In Section 3, we shall discuss the question of determining for which groups the isovariant Borsuk-Ulam theorem holds. In Section 4, we shall discuss the existence of an isovariant map between representations. Finally, in Section 5, other topics, in particular, the isovariant Borsuk-Ulam theorem for pseudofree $S^1$-actions and its converse are discussed.

## 2 Isovariant Borsuk-Ulam type theorems

We first recall some definitions and notations in transformation group theory. Let $G$ be a compact Lie group and $X$ a $G$-space. For any $x \in X$, the isotropy subgroup $G_x$

at $x$ is defined by $G_x = \{g \in G \mid gx = x\}$. We denote by $\mathrm{Iso}\, X$ the set of the isotropy subgroups. A subgroup of $G$ always means a closed subgroup. The notation $H \leq G$ means that $H$ is a subgroup of $G$, and $H < G$ means that $H$ is a proper subgroup of $G$. As usual $X^H$ denotes the $H$-fixed point set: $X^H = \{x \in X \mid hx = x \ (\forall h \in H)\}$. All $G$-equivariant maps ($G$-maps for short) are assumed to be continuous. A $G$-map $f : X \to Y$ is called *$G$-isovariant* if $f$ preserves the isotropy subgroups, i.e., $G_x = G_{f(x)}$ for all $x \in X$. The notion of isovariance was introduced by Palais [41] in order to study a classification problem on orbit maps of $G$-spaces. Moreover isovariant maps often play important roles in classification problems of $G$-manifolds or equivariant surgery theory, for example, see [7], [44], [50]. For the study of these maps from the viewpoint of homotopy theory, see [12].

As mentioned in the Introduction, the Borsuk-Ulam theorem can be stated in the context of equivariant topology or transformation group theory. In this context, there are very rich researches and results on Borsuk-Ulam type theorems, see, for example, [1], [3], [4], [9], [11], [13], [15], [16], [17], [19], [20], [26], [43]. Here we present the following generalization of the Borsuk-Ulam theorem for free $G$-spaces.

**Theorem 2.1** ([3], [37]). *Let $C_k$ be a cyclic group of order $k$. Let $X$ be an arcwise connected free $C_k$-space and $Y$ a Hausdorff free $C_k$-space. If there exists a positive integer $n$ such that $H_q(X; \mathbb{Z}/k) = 0$ for $1 \leq q \leq n$ and $H_{n+1}(Y/C_k; \mathbb{Z}/k) = 0$, then there is no continuous $C_k$-map from $X$ to $Y$. Here this homology means the singular homology.*

*Remark.* This result can be deduced from a more general result of [3] and therein the Borel cohomology and spectral sequence arguments are used. On the other hand, in [37], the Smith homology is used. The advantage of the latter method is that the proof is still valid in the category of definable sets with the $o$-minimal structure over a real closed field, see [36], [37].

Theorem 2.1 implies a well-known Borsuk-Ulam type theorem.

**Corollary 2.2** (mod $p$ Borsuk-Ulam theorem). *Let $C_p$ be a cyclic group of prime order $p$. Assume that $C_p$ acts freely on mod $p$ homology spheres $\Sigma^m$ and $\Sigma^n$. If there is a $C_p$-map $f : \Sigma^m \to \Sigma^n$, then $m \leq n$. In other words, if $m > n$, then there is no $C_p$-map from $\Sigma^m$ to $\Sigma^n$.*

In addition, one can see the following.

**Corollary 2.3.** *Let $S^1$ be a circle group. Assume that $S^1$ acts smoothly and fixed-point-freely on rational homology spheres $\Sigma^m$ and $\Sigma^n$. If there is an $S^1$-map $f : \Sigma^m \to \Sigma^n$, then $m \leq n$.*

*Proof.* One can take a large prime number $p$ such that $C_p$ $(\leq S^1)$ acts freely on $\Sigma^m$ and $\Sigma^n$, and that $\Sigma^m$ and $\Sigma^n$ are mod $p$ homology spheres. □

Thus Borsuk-Ulam type theorems are thought of as nonexistence results of $G$-maps. In this direction, we discuss the nonexistence of $G$-isovariant maps; namely, the isovariant Borsuk-Ulam theorem. We here recall a linear action or a homologically linear action on a (homology) sphere. Let $V$ be a real representation of $G$, i.e., $V$ is a (finite dimensional) real vector space on which $G$ acts linearly. Since every representation of compact Lie group $G$ is isomorphic to an orthogonal representation [2], we may suppose that the representation is orthogonal. Since, then, the $G$-action on $V$ preserves the standard metric, it induces linear $G$-actions on the unit sphere $SV$ and the unit disk $DV$. A $G$-manifold which is $G$-diffeomorphic to $SV$ [resp. $DV$] is called a linear $G$-sphere [resp. a linear $G$-disk].

A *homologically linear* action on a homology sphere is defined as follows. Let $G$ be a compact Lie group. Set

$$R_G = \begin{cases} \mathbb{Z}/|G| & \text{if } \dim G = 0 \\ \mathbb{Z} & \text{if } \dim G > 0. \end{cases}$$

Let $\Sigma$ be an $R_G$-homology sphere, i.e., $H_*(\Sigma; R_G) \cong H_*(S^m; R_G)$, where $m = \dim \Sigma$. Suppose that $G$ acts smoothly on $\Sigma$.

**Definition.**

(1) The $G$-action on $\Sigma$ is called *homologically linear* if for every subgroup $H$ of $G$, the $H$-fixed point set $\Sigma^H$ is an $R_G$-homology sphere or the empty set.

(2) The $G$-action on $\Sigma$ is called *semilinear* or *homotopically linear* if for every subgroup $H$ of $G$, the $H$-fixed point set $\Sigma^H$ is a homotopy sphere or the empty set. (Hence $\Sigma$ itself must be a homotopy sphere.)

(3) We call a smooth closed manifold $\Sigma$ with homotopically linear [resp. semilinear] $G$-action a homotopically linear [resp. semilinear] $G$-sphere.

Let $\mathcal{H}_G$ denote the family of homologically linear $G$-spheres and $\mathcal{S}_G$ the family of semilinear $G$-spheres. We also denote by $\mathcal{L}_G$ the family of linear $G$-spheres.

*Remark.* Clearly $\mathcal{L}_G \subset \mathcal{S}_G \subset \mathcal{H}_G$, but the converse inclusions are not true in general. For semilinear actions on spheres, see [29], [30], [32].

**Lemma 2.4.** *Let $\mathcal{F}_G = \mathcal{H}_G$, $\mathcal{S}_G$ or $\mathcal{L}_G$.*

(1) *Let $H$ be a subgroup of $G$. If $\Sigma \in \mathcal{F}_G$, then $\Sigma \in \mathcal{F}_H$ by restriction of the action.*

(2) *Let $H$ be a normal subgroup of $G$. If $\Sigma \in \mathcal{F}_G$, then $\Sigma^H \in \mathcal{F}_{G/H}$.*

*Proof.* (1) Since $\Sigma^K$, $K \leq H$, is an $R_G$-homology sphere (or the empty set), it is also an $R_H$-homology sphere (or the empty set).

(2) Since $(\Sigma^H)^{K/H} = \Sigma^K$ is an $R_G$-homology sphere (or the empty set), it is also an $R_{G/H}$-homology sphere (or the empty set). □

Now we state the isovariant Borsuk-Ulam theorem for homologically linear actions.

**Theorem 2.5** (Isovariant Borsuk-Ulam theorem)**.** *Let $G$ be a solvable compact Lie group. If there is a $G$-isovariant map $f : \Sigma_1 \to \Sigma_2$ between homologically linear $G$-spheres $\Sigma_i$, $i = 1, 2$, then the inequality*

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G$$

*holds.*

A convention: if $\Sigma_i^G$ is empty, then we set $\dim \Sigma_i^G = -1$. To prove the theorem, we make the following definition.

**Definition.** We say that $G$ has the IB-property in $\mathcal{F}_G$, where $\mathcal{F}_G = \mathcal{L}_G$, $\mathcal{S}_G$ or $\mathcal{H}_G$ if $G$ has the following property: If there is a $G$-isovariant map $f : \Sigma_1 \to \Sigma_2$, $\Sigma_i \in \mathcal{F}_G$, then the inequality

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G$$

holds.

We first show the following fact.

**Lemma 2.6** ([49], [31]).

(1) *Let $H$ be a normal subgroup of $G$. If $H$ and $G/H$ have the IB-properties in $\mathcal{F}_H$ and $\mathcal{F}_{G/H}$ respectively, then $G$ also has the IB-property in $\mathcal{F}_G$.*

(2) *Let $\mathcal{F}_G = \mathcal{S}_G$ or $\mathcal{L}_G$. If $G$ has the IB-property in $\mathcal{F}_G$, then $G/H$ also has the IB-property in $\mathcal{F}_{G/H}$.*

*Proof.* (1) Let $f : \Sigma_1 \to \Sigma_2$ be any $G$-isovariant map between $\Sigma_1$ and $\Sigma_2$ in $\mathcal{F}_G$. Then $\mathrm{res}_H f : \Sigma_1 \to \Sigma_2$ is $H$-isovariant and $f^H : \Sigma_1^H \to \Sigma_2^H$ is $G/H$-isovariant. It follows from Lemma 2.4 that $\Sigma_1, \Sigma_2 \in \mathcal{F}_H$ and $\Sigma_1^H, \Sigma_2^H \in \mathcal{F}_{G/H}$. By assumption, we have

$$\dim \Sigma_1 - \dim \Sigma_1^H \leq \dim \Sigma_2 - \dim \Sigma_2^H,$$

$$\dim \Sigma_1^H - \dim \Sigma_1^G \leq \dim \Sigma_2^H - \dim \Sigma_2^G.$$

Hence we obtain

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G.$$

(2) Suppose that $f : \Sigma_1 \to \Sigma_2$ is a $G/H$-isovariant map between $\Sigma_1$ and $\Sigma_2 \in \mathcal{F}_{G/H}$. Via the projection $G \to G/H$, the $G/H$-action lifts to a $G$-action. Hence $\Sigma_i$, $i = 1, 2$, are thought of as in $\mathcal{F}_G$ and then $f$ is a $G$-isovariant map. Thus we have $\dim \Sigma_1 - \dim \Sigma_1^{G/H} \leq \dim \Sigma_2 - \dim \Sigma_2^{G/H}$, since $\dim \Sigma_i^{G/H} = \dim \Sigma_i^G$. □

*Proof of Theorem* 2.5. We show that a solvable compact Lie group has the IB-property in $\mathcal{H}_G$. Suppose that $f : \Sigma_1 \to \Sigma_2$ is a $G$-isovariant map. Since $G$ is solvable, there is a normal series of closed subgroups:

$$1 = H_0 \lhd H_1 \lhd \cdots \lhd H_r = G$$

such that $H_i/H_{i-1}$, $1 \leq i \leq r$, is isomorphic to $C_p$ ($p$: some prime) or $S^1$.

By Lemmas 2.4 and 2.6, the proof is reduced to the cases of $C_p$ and $S^1$; moreover, the case of $S^1$ is also reduced to the case of $C_p$, since there exists some cyclic subgroup $C_p$ of $S^1$ such that $\Sigma_i^{S^1} = \Sigma_i^{C_p}$, $i = 1, 2$, in fact, $\Sigma_i \in \mathcal{H}_G$ has only finitely many orbit types [6], [18].

In the case of $C_p$, the proof proceeds as follows. Let $G = C_p$. Since $f$ is $G$-isovariant, it follows that $f(\Sigma_1 - \Sigma_1^G) \subset \Sigma_2 - \Sigma_2^G$. Set $N_i := \Sigma_i - \Sigma_i^G$. Since $\Sigma_i$ and $\Sigma_i^G$

are mod $p$ homology spheres, by Alexander duality, one can see that $N_i := \Sigma_i - \Sigma_i^G$ has the same mod $p$ homology groups as a sphere $S^{n_i}$, where $n_i = \dim \Sigma_i - \dim \Sigma_i^G - 1$. Since $G = C_p$ acts freely on $N_i$ and $f|_{N_1} : N_1 \to N_2$ is a $G$-map, it follows from Theorem 2.1 that $n_1 \leq n_2$, and hence

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G.$$

Thus $C_p$ has the IB-property in $\mathcal{H}_{C_p}$. □

This theorem implies Proposition 1.3.

*Proof of Proposition* 1.3. Let $f : V \to W$ be a $G$-isovariant map between representations. Let $V^{G\perp}$ denote the orthogonal complement of $V^G$ in $V$, and then $V$ decomposes as $V = V^G \oplus V^{G\perp}$. Similarly $W$ decomposes as $W = W^G \oplus W^{G\perp}$. Then the composition map $g := p \circ f \circ i : V^{G\perp} \to W^{G\perp}$ is a $G$-isovariant map, where $i : V^{G\perp} \to V$ is the inclusion and $p : W \to W^{G\perp}$ is the projection. Since $g^{-1}(0) = \{0\}$, $g$ induces a $G$-isovariant map $g_0 : V^{G\perp} \setminus \{0\} \to W^{G\perp} \setminus \{0\}$. By normalizing, one has a $G$-isovariant map $g_1 : S(V^{G\perp}) \to S(W^{G\perp})$. Since $G$ has the IB-property in $\mathcal{H}_G$, one has

$$\dim S(V^{G\perp}) + 1 \leq \dim S(W^{G\perp}) + 1,$$

which leads to the inequality

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

□

The following is obtained from Smith theory [6], [18].

**Corollary 2.7.** *Let $G$ be a finite $p$-group. Let $\Sigma_i$, $i = 1, 2$, be mod $p$ homology spheres with $G$-actions. If there is a $G$-isovariant map $f : \Sigma_1 \to \Sigma_2$, then*

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G$$

*holds.*

*Proof.* Smith theory shows that for every subgroup $H$, the $H$-fixed point set $\Sigma_i^H$ is a mod $p$ homology sphere or the empty set. Hence the $G$-action on $\Sigma_i$ is homologically linear and $\Sigma_i \in \mathcal{H}_G$. □

The singular set $N^{>1}$ of a $G$-manifold $N$ is defined by

$$N^{>1} = \bigcup_{1 \neq H \leq G} N^H.$$

The following is a variant of the isovariant Borsuk-Ulam theorem, which is a generalization of a result in [33].

**Corollary 2.8.** *Let $G$ be a finite group and $W$ a representation of $G$. Let $M$ and $N$ be homologically linear $G$-spheres and assume that $G$ acts freely on $M$. If there is a $G$-isovariant map $f : M \to N$, then the inequality*

$$\dim M + 1 \leq \dim N - \dim N^{>1}$$

*holds.*

*Proof.* Since $\dim N^{>1} = \max\{\dim N^H \mid 1 \neq H \leq G\}$, there is a subgroup $H \neq 1$ such that $\dim N^H = \dim N^{>1}$. Taking a cyclic subgroup $C_p \leq H$ of prime order, one has $\dim N^{C_p} = \dim N^{>1}$. By restricting to the $C_p$-action, it turns out that $f$ is a $C_p$-isovariant map. Hence, by the isovariant Borsuk-Ulam theorem, one has

$$\dim M + 1 \leq \dim N - \dim N^{C_p} = \dim N - \dim N^{>1}.$$

□

*Remark.* Not all finite groups can act freely on a (homology) sphere. For details, see [8], [23], [24], [25].

# 3 Which groups have the IB-property?

As seen in the previous section, a solvable compact Lie group has the IB-property in $\mathcal{F}_G$, i.e., the isovariant Borsuk-Ulam theorem holds in $\mathcal{F}_G$. In this section, we discuss the question: Which compact Lie groups have the IB-property in $\mathcal{F}_G$? First we consider the case of $\mathcal{F}_G = \mathcal{H}_G$ or $\mathcal{S}_G$. In this case, a complete answer is known.

**Theorem 3.1** (cf. [31]). *The following statements are equivalent.*

(1) *$G$ has the IB-property in $\mathcal{H}_G$.*