4.1 Decline-Type Mortality Models

4.1.1 The Proportional Hazard Model (PH)

The proportional hazard model (abbreviated as PH) is a simple model that expresses mortality improvement as *decline*. In the PH model, $\lambda_{x,t}$: the log hazard rate function at time t is expressed by

$$\lambda_{x,t} = \log \mu_{x,t} = a_x + k_t$$

where ax: the baseline logged hazard rates.

In the PH model, $\rho_{x,t}$: the rate of mortality improvement

$$\rho_{x,t} = -\frac{dk_t}{dt} = -k_t^{'}$$

is constant with respect to age. This is the differential form for this model. In this paper, we fit and numerically evaluate the models against the Japanese female mortality. We use

$$m_{x,t_c}, \qquad x = x_s(=25),\ldots, x_e(=110) \quad \text{and} \quad t_c = t_s(=1970),\ldots, t_e(=2007)$$

from the HMD (Human Mortality Database), where t_c is a calendar year. Here, we set a_x as the average log hazard rate in the entire period. Figure 10 shows the actual log hazard rates (λ_{x,t_c}) and the estimated rates with the PH model. We can observe that the estimated rates do not exhibit good fit particularly in the older age groups. Figure 11 shows the difference between the actual and estimated rates. From this graph, we can see that the actual values are higher than those of the model for age around from 60 to 80 in 1970, whereas these values are decreasing over time. However, opposite movement is observed for ages over 90. This is caused by the limitation of the PH model whereby the rate of mortality improvement is constant with respect to age.

Figure 10. Mortality rates (Actual and model, PH)

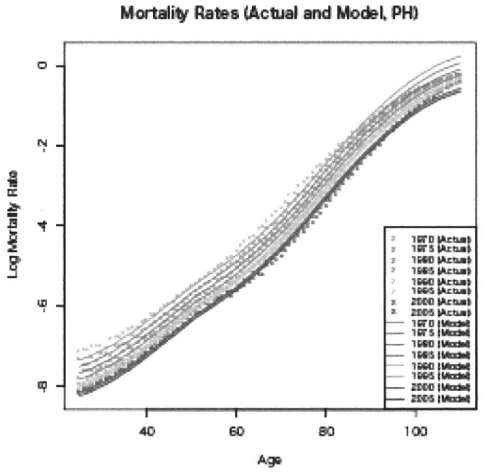
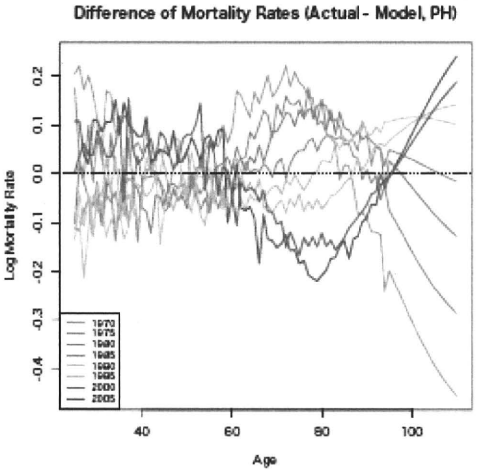


Figure 11. Difference of mortality rates (actual-Model, PH)



4.1.2 The Lee-Carter Model (LC)

The LC model is already defined in Section 3. It expresses mortality improvement as *decline* in a more general manner as compared with the PH model. Here, we set a_x as the average log hazard rate for the entire period. Figure 12 shows the actual log hazard rates (λ_{x,t_c}) and the estimated rates by the LC model. This figure illustrates that the fit with the actual values is fairly improved by using the LC model, due to its flexibility which admits different mortality improvement rates by age.

However, we can observe from Figure 13 that the difference between the actual and estimated rates exhibits a trend whereby the actual values are higher in younger age groups and lower in older age groups near the beginning and the end of the entire period, whereas the opposite is true around the middle of the period. The reason why this trend for the error components is observed is ascribed to the change in the age-specific mortality improvement rates over time. Therefore, we will next examine the $\rho_{x,t}$ functions for these two models.

Figure 12. Mortality rates (actual and model, LC)

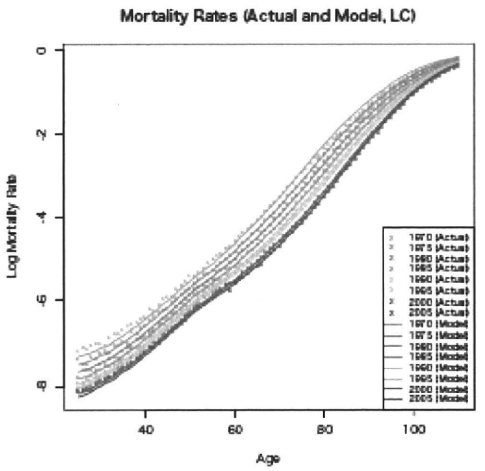
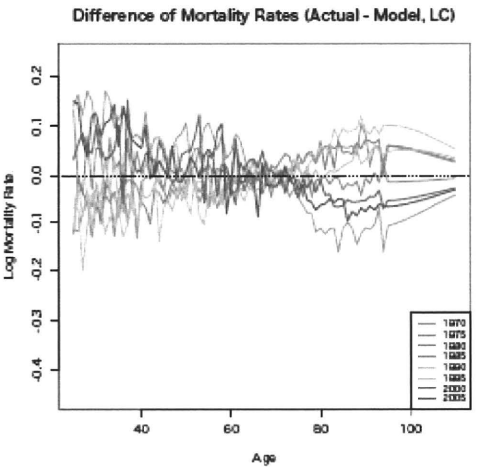


Figure 13. Difference of mortality rates (actual-model, lC)



Figures 14 and 15 show the ρ_{x,t_c} functions for the actual values and the estimated values for each of the two models. The blue lines show the ρ_{x,t_c} by the actual mortality rates. We can observe that most of the mortality improvement rates have mountain-shaped curves with peaks. In contrast, the mortality improvement rates under the PH model, expressed by the pink line, are horizontal. This difference in shape would be viewed as a cause that the estimates by the PH model are not well-fitted, as we observed before. The mortality improvement rate by the LC model, indicated with the green curves, has a peak like that of the actual value, and this improves the fit as we have seen before. However, the age distribution of the rates is fixed in the LC model, whereas it changes dynamically in the actual values. Thus, the actual age distribution of mortality improvement rates change over time and are not constant as in the LC model, and caused the propensity for the error in the LC model observed in Figure 13. We could see this result as a limitation when the mortality improvement is considered as *decline*.

Figure 14. Comparison of mortality improvement rates (1974-1989)

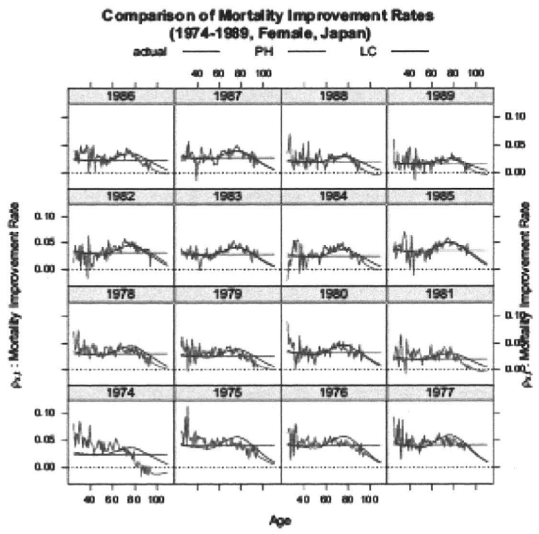
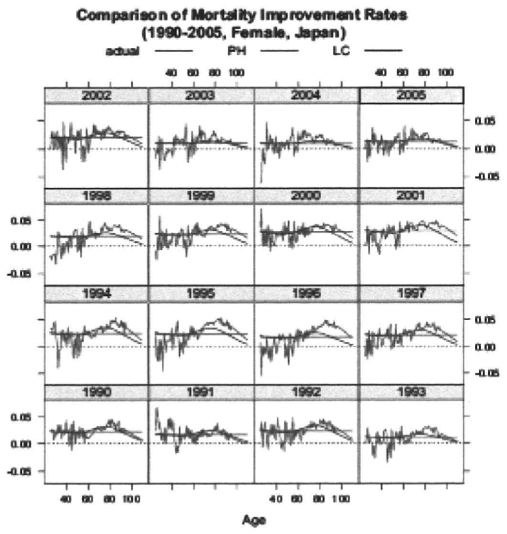


Figure 15. Comparison of mortality improvement rates (1990-2005)



4.2 Shift-Type Mortality Models

4.2.1 The Horizontal Shifting Model (HS)

Next, we discuss models that express mortality improvement through a *shift*. The simplest model for *shifting* would be one whereby the entire log hazard curve moves to the right-hand side. We can restate this model using the inverse function of log hazard mortality $v_{y,t}$, that is, the proportional hazard model for $v_{y,t}$.

This model that we call the horizontal shifting model (abbreviated as HS) here is formally expressed as follows:

$$v_{y,t} = a_y + k_t$$

In the differential form,

$$\tau_{y,t} = \frac{dk_t}{dt} = k_t^{'}$$

Parameter estimation for the HS model is completely identical to the PH models, except for adapting these procedures to v_{y,t_c} instead of λ_{y,t_c}. Figures 16 and 17 are the actual inverse mortality rates (v_{y,t_c}) and the estimated rates by the HS model, and the difference between the actual and the estimated. We can see that the performance of fitting by the HS model is much better than by the PH model, even though both have the same structure. For 1970, indicated with the light blue line, the actual values are higher in younger ages and lower in older ages, though the errors are not as high for other years.

Figure 16. Inverse mortality rates (actual and model, HS)

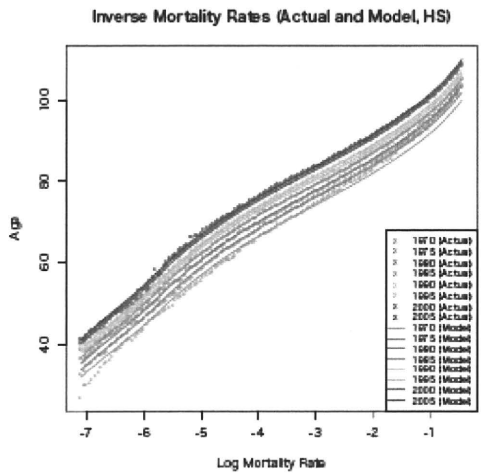
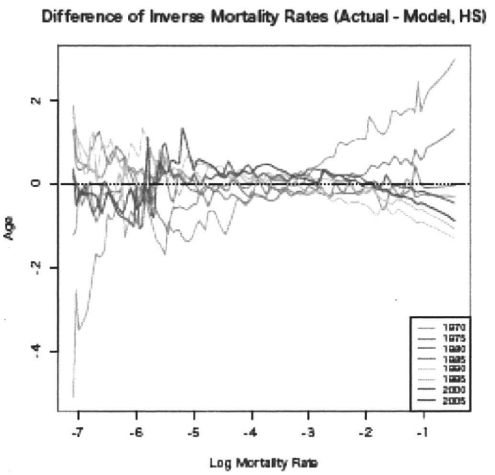


Figure 17. Difference of inverse mortality rates (actual and model, HS)



4.2.2 The Horizontal Lee-Carter Model (HL)

As we considered the LC model which admits a different amount of decline by age and provides a more general framework compared with the PH model, we can also consider the Lee-Carter model for $v_{y,t}$ which in turn supports a more general shifting feature. We call it the horizontal Lee-Carter model (abbreviated as HL).

$$v_{y,t} = a_y + k_t b_y$$

In the differential form,

$$\tau_{y,t} = \frac{dk_t}{dt} b_y = -k_t' b_y$$

Figures 16 and 17 are the actual inverse mortality rates (v_{y,t_c}) and the estimated rates under the HS model, and the difference between the actual and the estimated. We can see that the HL model seems to be improved compared to the HS model. However, it is also observed that the improvement between the *shift* pair is not as large as the *decline* pair. This means that relaxing the limitation, which the force of age increase in the HS model is restricted to the constant function, does not cause significant improvement of fit in the HL model. It could be explained by the difference in the shape of τ_{y,t_c}, the force of age increase. Figures 20 and 21 show the τ_{y,t_c} functions for the actual values and the estimated values by the two shifting models. We observe that the green curves, which correspond to τ_{y,t_c} by the HL model, are close to a horizontal line, which coincides with the force of age increase by the HS model shown in the pink lines. This fact endorses that the improvement between the *shift* pair is not as large as the decline pair.

Figure 18. Inverse Mortality Rates (actual and model, HL)

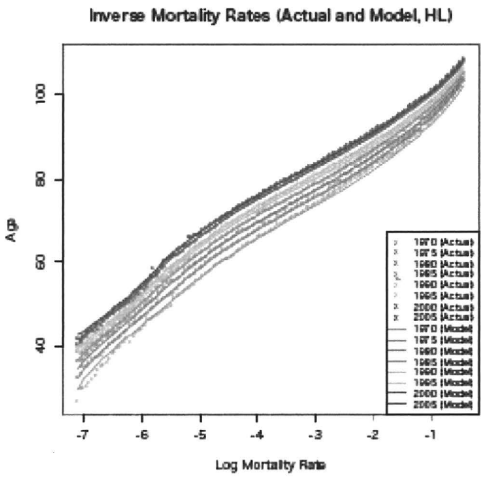
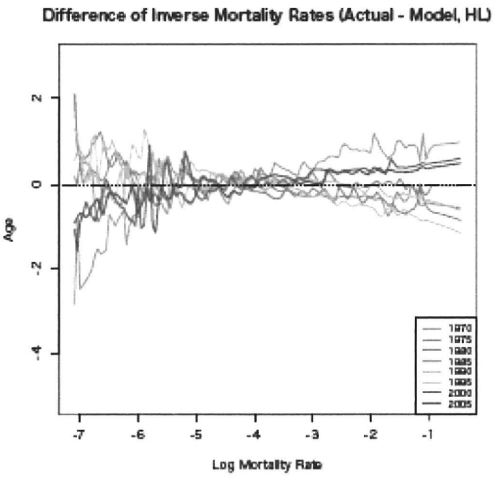


Figure 19. Difference of Inverse Mortality Rates (actual and model, HL)



However, from the observation of these figures, we have noticed that the blue lines for the actual τ_{y,t_c} for each year could be more well-modelled by a linear function of y, which has led us to the development of a new model: the linear difference model. We will define and examine this new model in the next section.

Figure 20. Comparison of the force of age increase by log mortality rate (1974-1989)

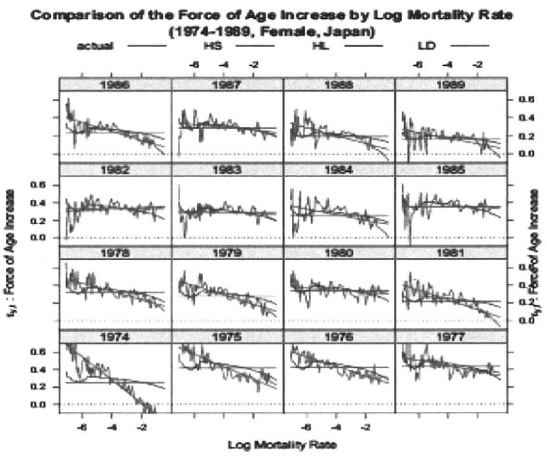
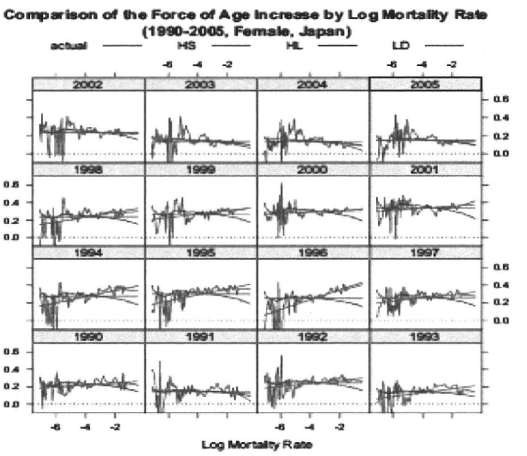


Figure 21. Comparison of the force of age increase by log mortality rate (1990-2005)



4.3 The Linear Difference Model (LD)

First, we describe the linear difference model (abbreviated as LD) in the continuous form as we did in other models. In the LD model, we assume that $\tau_{y,t}$ is a linear function of y for each t.

$$\tau_{y,t} = k_t^{'} + c_t^{'} y$$

This is the differential form. By integrating both sides with t, we obtain

$$v_{y,t} = k_t + c_t y + a_y$$

where a_y denotes a standard pattern of inverse log hazard rates.

Figures 22 and 23 are the actual inverse mortality rates and the estimated rates by the LD model, and the difference between the actual and the estimated. From these figures, we can observe that the LD model fits quite well with the actual values.

This is also confirmed from the observation of $\tau_{y,t}$ functions in Figures 20 and 21. We can observe that the linear assumption for τ_{y,t_c} in the LD model works better than in the other two models.

Figure 22. Inverse mortality rates (actual and model, LD)

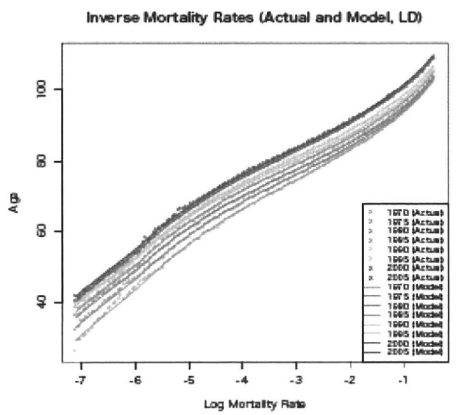
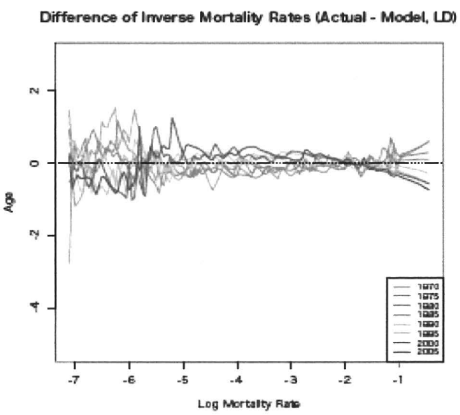


Figure 23. Difference of inverse mortality rate (actual and model, LD)



4.4 Comparison of the Models from a Statistical Viewpoint

In this section, we compare the LC and LD models from a statistical viewpoint to examine whether it is more plausible to understand the recent Japanese mortality as *declining* or *shifting*. Our approach is as follows.

1. The true mortality rates are assumed to be those that are estimated by models.

2. The number of deaths follows a binomial distribution $B(N_{x,t_c}, p_{x,t_c})$, where N_{x,t_c} : the number of the population and p_{x,t_c} : the death rate for age x and calendar year t_c.

3. N_{x,t_c} is approximated by the closest integer to E_{x,t_c} : exposure to risk.

Here, we took 0.01% as a critical value to construct the confidence intervals (CI), since $N_{x,t}$ would present too large value for the Japanese female population. Figure 24 shows the proportion where the log actual mortality rates are outside of the CIs for each age in the LC and LD models. This indicates that even though the proportions of LD are higher for certain ages, LD's performance would be considered as fairly better than LC's as a whole. This result suggests that *shift* is more strongly supported as recognition of the recent mortality improvement in Japan than *decline*.

Figure 24. proportion that log actual values are outside of CI (critical value=0.01%)

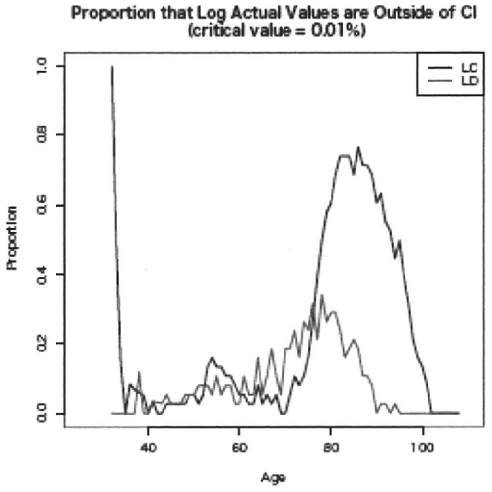
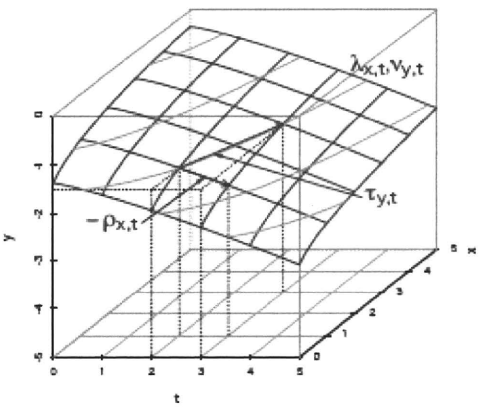


Figure 25. log mortality surface and two differential functions.



4.5 Differential Forms and Age-transformations

Next, we consider the relationship between differential forms and age-transformations, and discuss how the LD model is related to the age-transformation approach.

In section 1, we defined $\rho_{x,t}$ and $\tau_{y,t}$ on the log mortality surface *S*. Then, the vectors

$$\rho(x_0,t_0,y_0)=(0,1,-\rho_{x_0,t_0})$$

$$\tau(x_0,t_0,y_0)=(\tau_{y_0,t_0},1,0)$$

are tangent vectors on *S* as shown in Figure 25. Each tangent vector defines a tangent vector field on *S*. In general, an iso-transformed age map is defined by the projection of the integral curve induced by the tangent vector field onto a *X-T* plane. For example, the iso-transformed age map induced by ρ is an identity age-transformation, and one by τ is an age-transformation that identifies the ages that yield the same log hazard rates. If we define another tangent vector field on *S*, then another iso-transformed age map is induced. Therefore, a tangent vector field on *S* is considered as another representation of an age-transformation.

Let us recall that the LD model is defined by a differential form that is a modeling of $\tau_{y,t}$. Therefore, the LD model defines an age-transformation through the vector field interpretation with a tangent vector τ. This relationship relates the LD model to the age-transformation approach.

5. Concluding Remarks

In this paper, we examined and proposed a new method for mortality projection for Japan as an application of the age-transformation approach.

We considered which is more plausible to understand mortality improvement in Japan as decline or shift. First, we described the definitions of the proportional hazard model and the Lee-Carter model, which are *decline*-type models. Then, we introduced the horizontal shifting model and the horizontal Lee-Carter model, which are *shift*-type models corresponding to the two *decline* type ones. Next, we noticed that the actual τ_{y,t_c} for each year could be well-modelled by a linear function of y, and proposed the linear difference (LD) model. We observed that the LD model coincided quite well with the actual values.

Then, we compared the LC and LD models from a statistical viewpoint to examine whether it is more plausible to understand the recent Japanese mortality as a *decline* or *shift*. We observed that LD's performance would be considered advantageous over LC's as a whole. This result suggests that *shift* is more strongly supported as recognition of the recent mortality improvement in Japan than *decline*.

Finally, we considered the relationship between differential forms and age-transformations, and discussed how the LD model is related to the age-transformation approach. In general, an iso-transformed age map is defined by the projection of the integral curve induced by the tangent vector field onto $X - T$ plane. Therefore, a tangent vector field on S is considered as another representation of an age-transformation. The LD model is defined by a differential form that is a modelling of $\tau_{y,t}$. Therefore, the LD model defines an age-transformation through vector fields interpretation with tangent vector $\boldsymbol{\tau}$. This relationship relates the LD model to the age-transformation approach.

In this paper, we confirmed that the LD model is efficient, although we noted further points that should be examined. First, we discussed only the adult mortality model here, whereas the entire age model should be developed. Second, we focused on the modelling of the actual values in this paper, whereas we should consider how to project the parameters. These points should be studied in future.

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