dimensional) real vector spaces V and W. If there exists a G-isovariant map from V to W, then the inequality

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds. Here V^G denotes the G-fixed point sets of V.

In other words, if $\dim V - \dim V^G > \dim W - \dim W^G$, the there is no isovariant map from V to W. In this note, we discuss an existence problem of isovariant maps between G-representation spaces, which is considered as the converse of the isovariant Borsuk-Ulam theorem. In [11], we discussed it when G is an finite abelian p-group or a certain finite cyclic group. In this note, we shall discuss it when G is a certain dihedral group and show an existence result of isovariant maps under some dimensional conditions. The main result is Theorem 4.1 in section 4.

2. Basic facts of isovariant maps

In this section, G is a compact Lie group. Notation $H \leq G$ means that H is a closed subgroup of G and a subgroup means a closed subgroup.

We recall some basic facts for isovariant maps. Let $f : X \to Y$ be a G-map. Throughout this article, all maps are understood to be continuous. Let H be a subgroup of G. By restriction of the G-action to the H-action, an H-map $\mathrm{Res}_H f : \mathrm{Res}_H X \to \mathrm{Res}_H Y$ is defined, and by restriction of f to the H-fixed point set X^H, an $N_G(H)/H$-map $f^H : X^H \to Y^H$ is defined, where $N_G(H)$ denotes the normalizer of H in G. Let H be a normal. subgroup of G. Let X, Y be G/H-spaces and $f : X \to Y$ a G/H-map. Via the projection $p : G \to G/H$, X and Y are regarded as G-spaces, denoted by $\mathrm{Inf}_{G/H}^G X$ and $\mathrm{Inf}_{G/H}^G Y$ respectively, and f is regarded as a G-map, denoted by $\mathrm{Inf}_{G/H}^G f$. These are called the *inflation* of a G/H-space or a G/H-map. We often omit the symbols Res_H and $\mathrm{Inf}_{G/H}^G$ for simplicity if there is no misunderstanding in context.

In [11] we have shown the following.

Lemma 2.1. *The following hold.*

(1) *If f is G-isovariant, then $\mathrm{Res}_H f$ is H-isovariant for any $H \leq G$.*
(2) *Let H be a normal subgroup. If f is G-isovariant, then f^H is G/H-isovariant.*
(3) *Let H be a normal subgroup. If $f : X \to Y$ is G/H-isovariant, then $\mathrm{Inf}_{G/H}^G f$ is G-isovariant.*
(4) *If $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ are G-isovariant, then $f \times g : X_1 \times X_2 \to Y_1 \times Y_2$ is G-isovariant under the diagonal action.*

(5) *If* $f : X_1 \to Y_1$ *and* $g : X_2 \to Y_2$ *are* G*-isovariant, then so is* $f * g : X_1 * X_2 \to Y_1 * Y_2$*, where* $*$ *means join, in particular, the cone of* f*,* $Cf : CX_1 \to CY_1$*, is* G*-isovariant.*

(6) *If* $f : X \to Y$ *and* $g : Y \to Z$ *are* G*-isovariant, then so is* $g \circ f : X \to Z$.

(7) *If* $f : X \to Y$ *is* H*-isovariant, then* $G \times_H f : G \times_H X \to G \times_H Y$ *is* G*-isovariant.*

We may assume that a real G-representation V is an orthogonal G-representation without loss of generality, because every real representation is isomorphic to an orthogonal representation [17]. Then, the standard inner product on V is G-invariant, and so the unit disk and the unit sphere are G-invariant submanifolds, denoted by SV and DV respectively. Moreover $V^{G\perp}$ denotes the orthogonal complement of V^G in V, which is a subrepresentation of V.

The main results in [11] are not affected, but Lemma 2.2 of [11] is unfortunately missing an assumption. We correct it as follows.

Lemma 2.2. *Let* V*,* W *be* G*-representations. For statements* (1)–(6) *below* (1) *implies* (2) *and statements* (2)–(6) *are equivalent. Furthermore if* $\mathrm{Iso}\, SV \subset \mathrm{Iso}\, SW$*, then all statements are equivalent, where* $\mathrm{Iso}\, SV$ *denotes the set of isotropy subgroups of* SV*.*

(1) *There exists a* G*-isovariant map from* SV *to* SW*.*

(2) *There exists a* G*-isovariant map from* V *to* W*.*

(3) *There exists a* G*-isovariant map from* $S(V^{G\perp})$ *to* $S(W^{G\perp})$*.*

(4) *There exists a* G*-isovariant map from* $V^{G\perp}$ *to* $W^{G\perp}$*.*

(5) *There exists a* G*-isovariant map from* DV *to* DW*.*

(6) *There exists a* G*-isovariant map from* $D(V^{G\perp})$ *to* $D(W^{G\perp})$*.*

Proof. (1) $\Rightarrow$ (2): Take the open cone of an isovariant map $f : SV \to SW$.

(2) $\Rightarrow$ (4): Let $f : V \to W$ be an isovariant map. The inclusion $i : V^{G\perp} \to V$ is clearly G-isovariant, and the projection $p : W = W^{G\perp} \oplus W^G \to W^{G\perp}$ is also G-isovariant, since G acts trivially on W^G. Hence the composite map $p \circ f \circ i : V^{G\perp} \to W^{G\perp}$ is G-isovariant.

(4) $\Rightarrow$ (3): Let $f : V^{G\perp} \to W^{G\perp}$ be an isovariant map. Since $(V^{G\perp})^G = (W^{G\perp})^G = 0$, we have $f^{-1}(0) = \{0\}$, and hence a G-isovariant map $g : S(V^{G\perp}) \to S(W^{G\perp})$ can be defined by $g(x) = f(x)/\|f(x)\|$.

(3) $\Rightarrow$ (6): Taking the cone of an isovariant map $f : S(V^{G\perp}) \to S(W^{G\perp})$, we obtain a G-isovariant map $\tilde{f} : D(V^{G\perp}) \to D(W^{G\perp})$.

(6) $\Rightarrow$ (5): Since any map $g : D(V^G) \to D(W^G)$ is G-isovariant, taking product, we obtain a G-isovariant map

$$f \times g : D(V) \cong D(V^{G\perp}) \times D(V^G) \to D(W) \cong D(W^{G\perp}) \times D(W^G).$$

(5) $\Rightarrow$ (2): Let $f : DV \to DW$ be a G-isovariant map. Define $g : DV \to DW$ by $g(x) = f(x)/2$, then g is G-isovariant and maps DV to the interior $\operatorname{Int} DW$ of DW. Hence we obtain a G-isovariant map $g|_{\operatorname{Int} DV} : V \cong_G \operatorname{Int} DV \to W \cong_G \operatorname{Int} DW$.

When $\operatorname{Iso} SV \subset \operatorname{Iso} SW$, the implication (3) $\Rightarrow$ (1) is shown as follows. Let $f : S(V^{G\perp}) \to S(W^{G\perp})$ be an isovariant map. If $SV^G = \emptyset$, then $SV = S(V^{G\perp})$, and hence $i \circ f : SV \to S(W^{G\perp}) \subset SW$ is isovariant. If $SV^G \neq \emptyset$, then $SW^G \neq \emptyset$, since $\operatorname{Iso} SV \subset \operatorname{Iso} SW$. Take any map $g : SV^G \to SW^G$. The join of f and g gives an isovariant map from SV to SW. Thus the proof is complete. □

3. The existence problem of isovariant maps

In this section, G is a finite group.

Definition. We say that G has the *IB-property* over $\mathbb{R}$ [resp. $\mathbb{C}$]. if the isovariant Borsuk-Ulam theorem holds for orthogonal [resp. unitary] G-representations; namely, the inequality $\dim V - \dim V^G \leq \dim W - \dim W^G$ holds for every pair (V, W) of G-representations such that there is a (continuous) G-isovariant map from V to W.

As being proved below, G has the IB-property over $\mathbb{R}$ if and only if G has the IB-property over $\mathbb{C}$, and therefore we say for simplicity that G has the IB-property.

Proposition 3.1. *The following statements are equivalent.*

(1) *G has the IB-property over $\mathbb{R}$*

(2) *G has the IB-property over $\mathbb{C}$*

Proof. (1) $\Rightarrow$ (2): Suppose that $f : V \to W$ is a G-isovariant map between unitary representations. By restriction of the ground field $\mathbb{C}$ to $\mathbb{R}$, a G-isovariant map $\operatorname{Res}_{\mathbb{R}} f : \operatorname{Res}_{\mathbb{R}} V \to \operatorname{Res}_{\mathbb{R}} W$ between orthogonal representations is obtained, and hence the inequality $\dim V - \dim V^G \leq \dim W - \dim W^G$ holds.

(2) $\Rightarrow$ (1): Suppose that $f : V \to W$ is a G-isovariant map between orthogonal representations. Taking a direct sum, we obtain an isovariant map $f \oplus f : V \oplus V \to W \oplus W$. Since $V \oplus V$ has a complex structure, in fact, $V \oplus V \cong V \otimes \mathbb{C}$ for some complex structure on $V \oplus V$, we obtain a continuous isovariant map $f \oplus f : V \otimes \mathbb{C} \to W \otimes \mathbb{C}$. Hence the inequality $\dim V - \dim V^G \leq \dim W - \dim W^G$ holds. □

By Theorem 1.1, every finite solvable group has the IB-property. Wasserman [18] shows that a finite groups satisfying the prime condition (e.g., A_5, S_5 etc.) also have the IB-property, and Nagasaki [9] says that a weaker version of the isovariant Borsuk-Ulam theorem holds for an arbitrary compact Lie group.

Remark. The author, however, does not know an example of not having the IB-property.

Let G be a finite solvable group, and V and W G-representations. Suppose that there exists a G-isovariant map from V to W. For any pair of subgroups $H \triangleleft K$ (H is normal in K), the restriction of f to the H-fixed point sets yields a K/H-isovariant map $f^H : V^H \to W^H$. Since K/H is also solvable, the isovariant Borsuk-Ulam theorem shows that

$(C_{V,W})$: $\dim V^H - \dim V^K \leq \dim W^H - \dim W^K$ for any pair $H \triangleleft K$.

Moreover the pair (V, W) must obviously satisfy

$(I_{V,W})$: $\operatorname{Iso} V \subset \operatorname{Iso} W$,

Definition. We say that a finite solvable group G has the *complete IB-property* over $\mathbb{R}$ [resp. $\mathbb{C}$] if for every pair (V, W) of orthogonal [resp. unitary] G-representations satisfying conditions $(C_{V,W})$ and $(I_{V,W})$, there exists a G-isovariant map from V to W.

In [11] we show that if G is nilpotent, then $(C_{V,W})$ implies $(I_{V,W})$. In the case, $(I_{V,W})$ can be removed from the above definition. Note also that if G has the complete IB-property over $\mathbb{R}$, then G has the complete IB-property over $\mathbb{C}$; however, the author does not know whether the converse of this property holds. We have proposed the following question in [11].

Question. Which finite solvable groups have the complete IB-property over $\mathbb{R}$ (or $\mathbb{C}$)?

4. Results and proofs

Concerning the question, we have shown [11] that the following groups have the complete IB-property over $\mathbb{R}$.

(1) abelian p-groups,
(2) $C_{p^m q^n}$, the cyclic group of order $p^m q^n$ ($m \geq 1$, $n \geq 1$),
(3) C_{pqr}, the cyclic group of order pqr.

Here p, q, r are distinct primes.

Remark. S. Kôno also obtained that $C_{p^m q^n}$ has the complete IB-property over $\mathbb{C}$.

We give new examples of having the complete IB-property over $\mathbb{R}$ and of not having the complete IB-property over $\mathbb{R}$.

Theorem 4.1. *Let D_n denote the dihedral group of order $2n$ $(n \geq 3)$.*

(1) *D_3, D_4 and D_6 have the complete IB-property over $\mathbb{R}$.*
(2) *Every D_n $(n \neq 3, 4, 6)$ does not have the complete IB-property over $\mathbb{R}$.*

Remark. Since $D_1 \cong C_2$ and $D_2 \cong C_2 \times C_2$, these also have the complete IB-property over $\mathbb{R}$ by the previous result.

Remark. In [12], the author announced that some dihedral groups have the the complete IB-property over $\mathbb{R}$, but unfortunately it is incorrect.

We first recall the subgroup structure of D_n. Let

$$D_n = \langle a, b \,|\, a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle.$$

There is the normal cyclic subgroup $C_m = \langle a^{n/m} \rangle$ of D_n for every divisor m of n, and there are n/m dihedral subgroups $\langle a^{n/m}, b \rangle, \langle a^{n/m}, b^2 \rangle, \ldots, \langle a^{n/m}, a^{n/m-1}b \rangle$ containing C_m as a subgroup of index 2. If n/m $(= [D_n : D_m])$ is odd, then these are all conjugate in D_n. We take $D_m = \langle a^{n/m}, b \rangle$ as a representative of their conjugacy class. If n/m is even, then there are two conjugacy classes whose representatives are $D_m = \langle a^{n/m}, b \rangle$ and $D'_m = \langle a^{n/m}, ab \rangle$.

We next recall the orthogonal irreducible representations of D_n from [17]. These are described as follows. Let $T_k = \mathbb{C}\,(= \mathbb{R}^2)$, $1 \leq k < n/2$, be the D_n-representation on which D_n acts by $a \cdot z = \xi^k z$, $b \cdot z = \bar{z}$, $z \in S_k$, where $\xi = \exp(2\pi\sqrt{-1}/n)$. These T_k are all (nonisomorphic) 2-dimensional irreducible representation. Other irreducible representation are 1-dimensional. If n is odd, then there are two 1-dimensional irreducible representations $\mathbb{R}_{++}$ and $\mathbb{R}_{+-}$, where $\mathbb{R}_{++}$ is the trivial representation and $\mathbb{R}_{+-}$ $(= \mathbb{R})$ is a representation on which the action is given by $a \cdot x = x$, $b \cdot x = -x$, $x \in \mathbb{R}$.

If n is even, in addition to $\mathbb{R}_{++}$ and $\mathbb{R}_{+-}$, there are more two 1-dimensional irreducible representations $\mathbb{R}_{-+}$ and $\mathbb{R}_{--}$. The actions on $\mathbb{R}_{-+}$ and $\mathbb{R}_{--}$ are given by $a \cdot x = -x$, $b \cdot x = x$, and by $a \cdot x = -x$, $b \cdot x = -x$, respectively. Note, then, that

$$\text{Ker } T_k = C_{(k,n)}, \ \text{Ker } \mathbb{R}_{+-} = C_n, \ \text{Ker } \mathbb{R}_{-+} = D_{n/2}, \ \text{Ker } \mathbb{R}_{--} = D'_{n/2}.$$

Note also that

$$\text{Iso } T_k = \{D_n, \langle a^{n/(k,n)}, a^t b\rangle, \langle a^{n/(k,n)}\rangle \mid 0 \le t \le n-1\},$$

$$\text{Iso } \mathbb{R}_{+-} = \{D_n, C_n\}, \quad \text{Iso } \mathbb{R}_{-+} = \{D_n, D_{n/2}\}, \quad \text{Iso } \mathbb{R}_{--} = \{D_n, D'_{n/2}\}.$$

We first show (1) of Theorem 4.1 in the case of $n = 3$, 4.

Proposition 4.2. *Suppose $n = 3$, 4. Given D_n-representations V and W satisfying condition $(C_{V,W})$, there exists a D_n-isovariant map from V to W.*

Proof. We first consider the case of $n = 3$. There is only one 2-dimensional irreducible representation T_1. By Lemma 2.2, we may assume that $V^{D_n} = W^{D_n} = 0$. Hence we may put

$$V = a_1 T_1 \oplus a_2 \mathbb{R}_{+-}$$

and

$$W = b_1 T_1 \oplus b_2 \mathbb{R}_{+-},$$

where a_i and b_i are non-negative integers, and $a_1 T_1$ denotes the direct sum of a_1 copies of T_1. The inequalities $(C_{V,W})$ for pairs $(1, C_3)$ and (C_3, D_6) lead to inequalities $a_1 \le b_1$ and $a_2 \le b_2$. Hence there is a natural inclusion from V to W, which is an isovariant map.

Next consider the case $n = 4$. In this case, there is only one 2-dimensional irreducible representation T_1. By a similar argument, we may put

$$V = a_1 T_1 \oplus a_2 \mathbb{R}_{+-} \oplus a_3 \mathbb{R}_{-+} \oplus a_4 \mathbb{R}_{--}$$

and

$$W = b_1 T_1 \oplus b_2 \mathbb{R}_{+-} \oplus b_3 \mathbb{R}_{-+} \oplus b_4 \mathbb{R}_{--}.$$

The inequalities $(C_{V,W})$ for pairs $(1, C_2)$, (C_4, D_4), (D_2, D_4), and (D'_2, D_4) lead us to inequalities $a_i \le b_i$, $i = 1, 2, 3, 4$. Hence there is a natural inclusion from V to W, which is an isovariant map. □

We next discuss the case of $n = 6$.

Proposition 4.3. *Given D_6-representations V and W satisfying condition $(C_{V,W})$, there exists a D_6-isovariant map from V to W.*

Proof. There are two 2-dimensional irreducible representations T_1 and T_2. By a similar argument as above, we may put

$$V = a_1 T_1 \oplus a_2 T_2 \oplus a_3 \mathbb{R}_{+-} \oplus a_4 \mathbb{R}_{-+} \oplus a_5 \mathbb{R}_{--}$$

and

$$W = b_1T_1 \oplus b_2T_2 \oplus b_3\mathbb{R}_{+-} \oplus b_4\mathbb{R}_{-+} \oplus b_5\mathbb{R}_{--}.$$

The following is a table for dimensions of H-fixed point sets of irreducible D_6-representations.

$\dim X^H$	1	D_1	D'_1	C_2	D_2	C_3	D_3	D'_3	C_6	D_6
T_1	2	1	1	0	0	0	0	0	0	0
T_2	2	1	1	2	1	0	0	0	0	0
$\mathbb{R}_{+-}$	1	0	0	1	0	1	0	0	1	0
$\mathbb{R}_{-+}$	1	1	0	0	0	1	1	0	0	0
$\mathbb{R}_{--}$	1	0	1	0	0	1	0	1	0	0

Considering pairs (C_2, C_6), (C_6, D_6), (D_3, D_6) and (D'_3, D_6), we obtain inequalities

$$a_i \le b_i \quad \text{for} \quad i = 2, 3, 4, 5$$

from condition $(C_{V,W})$. Considering pairs (D_1, D_2), (D'_1, D_2) and $(1, C_3)$, we also obtain inequalities

$$a_1 + a_4 \le b_1 + b_4, \quad a_1 + a_5 \le b_1 + b_5, \quad a_1 + a_2 \le b_1 + b_2.$$

If $a_1 \le b_1$, then there is a natural inclusion from V to W, which is an isovariant map.

We now assume that $a_1 > b_1$. The above inequalities shows

$$0 \le a_1 - b_1 \le b_2 - a_2, \quad 0 \le a_1 - b_1 \le b_4 - a_4, \quad 0 \le a_1 - b_1 \le b_5 - a_5.$$

Decompose V and W into

$$V = V' \oplus V'' \quad \text{and} \quad W = W' \oplus W'',$$

where

$$V' = (a_1 - b_1)T_1,$$

$$V'' = b_1T_1 \oplus a_2T_2 \oplus a_3\mathbb{R}_{+-} \oplus a_4\mathbb{R}_{-+} \oplus a_5\mathbb{R}_{--}$$

and

$$W' = (b_2 - a_2)T_2 \oplus (b_4 - a_4)\mathbb{R}_{-+} \oplus (b_5 - a_5)\mathbb{R}_{--},$$

$$W'' = b_1T_1 \oplus a_2T_2 \oplus b_3\mathbb{R}_{+-} \oplus a_4\mathbb{R}_{-+} \oplus a_5\mathbb{R}_{--}.$$

Since $a_3 \le b_3$, there is a natural inclusion from V'' to W''. Therefore It suffices to show the existence of an isovariant map from V' to W'. To show this, it suffices to show that there exists an isovariant map

$$f : T_1 \to T_2 \oplus \mathbb{R}_{-+} \oplus \mathbb{R}_{--},$$

since $a_1 - b_1 \leq a_i - b_i$, $i = 2, 4, 5$. Indeed this is proved as follows. Set $U = T_2 \oplus \mathbb{R}_{-+} \oplus \mathbb{R}_{--}$. The dimensions of H-fixed point sets are the following.

(H)	1	D_1	D_1'	C_2	D_2	C_3	D_3	D_3'	C_6	D_6
$\dim U^H$	4	2	2	2	1	2	1	1	0	0

By Lemma 2.2, it suffices to show that there exists an isovariant map $f : ST_1 \to SU$. It is easily checked that ST_1 is decomposed into $ST_1 = (G/\langle b\rangle \coprod G/\langle ab\rangle) \bigcup_\phi G \times I$ as a G-CW complex, where $G = D_6$ and I is the interval $[0,1]$. The attaching map $\phi : G \times \{0,1\} \to G/\langle b\rangle \coprod G/\langle ab\rangle$ is equivariantly defined by $\phi(1 \times 0) = \langle b\rangle \in G/\langle b\rangle$ and $\phi(1 \times 1) = \langle ab\rangle \in G/\langle ab\rangle$. On the other hand, SU has the 1-dimensional singular set

$$SU^{>1} = \bigcup_{0 \leq t \leq n-1} SU^{\langle a^t b\rangle} \bigcup SU^{C_2} \bigcup SU^{C_3}.$$

and its complement $SU \backslash SU^{>1}$ is path-connected. Note that G acts freely on $SU \backslash SU^{>1}$. Take points y, $z \in SU$ such that $G_y = \langle b\rangle$ and $G_y = \langle ab\rangle$. Then an isovariant map $f_0 : G/\langle b\rangle \coprod G/\langle ab\rangle \to SU$ is defined by $f_0(g\langle b\rangle) = gy$ and $f_0(g\langle ab\rangle) = gz$. Since $SU \setminus SU^{>1}$ is path-connected, one can take a path $\varphi : I \to SU$ such that $\varphi(0) = y$, $\varphi(1) = z$, and $\varphi(I \setminus \{0,1\}) \subset SU \setminus SU^{>1}$. Using this map, one can equivariantly extends f_0 to a map $f : ST_1 \to SU$, which is an isovariant map. Thus the proof is complete. □

Finally we discuss (2) of Theorem 4.1.

Proposition 4.4. *Suppose $n \neq 1, 2, 3, 4, 6$. Let k and l be distinct positive integers less than $n/2$ with $(k, n) = (l, n) = 1$. Then there is no D_n-isovariant map from T_k to T_l.*

Proof. Suppose that there is a D_n-isovariant map from T_k to T_l for some k and l. Lemma 2.2 shows that there is a D_n-isovariant map $f : ST_k \to ST_l$. Note that $ST_k^{>1} = ST_l^{>1} = \{\exp(\pi t\sqrt{-1}/n) \,|\, 0 \leq t \leq n-1\}$. Take $x = 1$ and $y = \exp(\pi\sqrt{-1}/n)$, then the isotropy subgroup G_x at x in ST_k is $\langle b\rangle$, and also $G_y = \langle a^s b\rangle$ in ST_k, where s is a positive integer with $ks \equiv 1 \bmod n$. Since $ST_l^{\langle b\rangle} = \{\pm 1\} \subset \mathbb{C}$, it follows that $f(1) = \pm 1$. Composing, if necessary, the antipodal map $z \mapsto -z$ with f, we may assume $f(1) = 1$. Let A is the shorter arc joining x with y in ST_k. Since every point of the interior of A has the trivial isotropy subgroup, it follows that $f(A \setminus \{x, y\})$ is contained in $ST_l \setminus ST_l^{>1}$; hence $f(y)$ must be y or $\bar{y}$. However the isotropy subgroup at y (resp. $\bar{y}$) in ST_l is equal to $\langle a^r b\rangle$ (resp. $\langle a^{-r} b\rangle$), where r is a positive integer with

$lr \equiv 1 \bmod n$, but it is not equal to $\langle a^s b \rangle$, since $l \not\equiv \pm k \bmod n$. This contradicts the isovariance of f. □

From this proposition, (2) of Theorem 4.1 is easily seen; in fact, every pair (T_k, T_l) as above satisfies conditions (C_{T_k,T_l}) and (I_{T_k,T_l}). Thus the proof is complete.

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