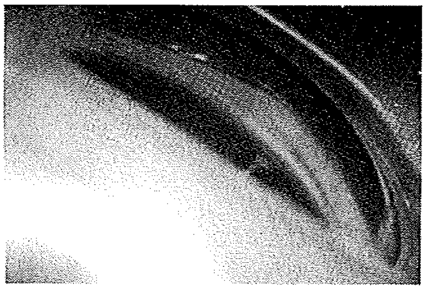

図5　隅角癒着解離術

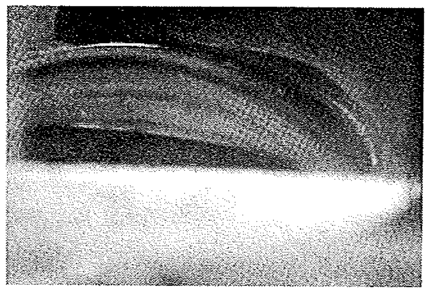

図6　線維柱帯切開術

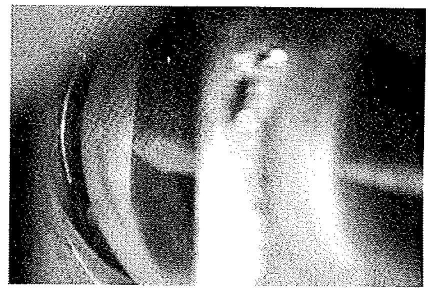

図7　プローブ回転が困難な際の手技

に房水が抜けることで前房が浅くなるのを防ぐために，可能な限り前房水のみならず後房水も抜いたうえで粘弾性物質に置換しておく必要がある．浅前房での無理な隅角操作は角膜内皮障害のリスクが高まるのみならず，術後の炎症を惹起しやすくなることから再癒着の頻度が高くなる．

2．線維柱帯切開術（trabeculotomy）（図6）

線維柱帯切開術の成功のためにはプローブの確実なSchlemm管内挿入が必須である．そのために森ゴニオレンズを用いて隅角側からプローブ挿入部を観察し，線維柱帯を通して視認できるプローブの金属光沢を確認することによって，早期穿孔や上脈絡膜腔への迷入を未然に防ぐことができる．また，症例によってはプローブの曲率とSchlemm管の曲率がぴったり合致，もしくは線維柱帯部の抵抗が非常に強いためにプローブの回転が困難な場合がある．このような場合に隅角側からプローブ先端部に向かって切開を加える（partial goniotomy，図7）ことによって，プローブの先端を前房内に出すことができ，容易にプローブの回転を完成させることが可能である．

3．その他の手術

濾過胞再建術の際のレクトミーのwindowの状態のチェック，隅角を鈍的に擦過除去するgoniocurettage[3]，線維柱帯を高周波によって切除するtrabectome[4]，iStentとよばれるSchlemm管-前房間のマイクロステント[5]など，隅角マイクロサージェリーを行う際にも森ゴニオレンズを使用することができると考えられる．

おわりに

森ゴニオレンズは，眼球や頭部を傾斜させることなく全周の隅角を観察・操作可能な手術用隅角鏡として有用である．

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角膜移植後の緑内障はこう治す

How to Treat Glaucoma after Corneal Transplantation

森　和彦*

はじめに

はたして緑内障とは治すことができる疾患であろうか．それは「治る」の言葉の定義次第である．「治る」を「完全に健康な元の状態に戻ること」とするなら，障害された神経を再生させることは現在の医学でもまだ不可能であるため，緑内障は不治の病ということになる．「治る」を「それ以上の進行を食い止めること」と定義するならば，現在の緑内障治療はすべからくこれを目標としており，かなりの症例で実行が可能となっている．しかし実際には眼圧を十分に下降させても進行を食い止めきれない症例が存在しており，「悪化のスピードを遅らせること」で何とか患者さんに納得してもらうしかない場合も多い．

一方，近年の角膜移植術の進歩に伴い，数多くの難治性眼表面疾患を「治す」ことが可能となってきた．さらにその移植方法も従来の角膜全層移植や表層移植のみならず，培養上皮シートを用いた角膜上皮移植やさらにはDSAEK (Descemet's stripping automated endothelial keratoplasty) に代表される内皮シート移植などのような角膜のパーツ移植へと進化してきている．このような進歩の背景にはLASIK (laser *in situ* keratomileusis) などの屈折矯正手術の普及のみならず，シクロスポリンに代表される免疫抑制療法の発展が大きく寄与している．今では眼類天疱瘡やStevens-Johnson症候群，急性期化学外傷などのように従来は禁忌とされてきた疾患に対しても積極的に外科的治療が施され，これらの疾患

図1　化学外傷による瘢痕性眼表面疾患

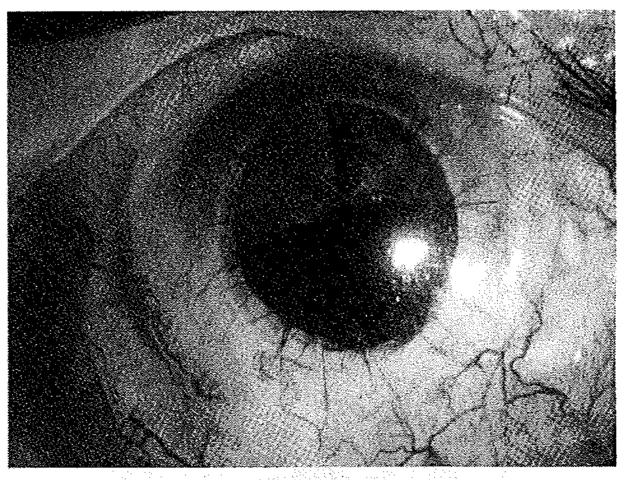

図2　図1と同一症例の羊膜移植を併用した眼表面再建術後

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が「治る」疾患へと変貌してきている（図1，2）．

I 角膜移植後の合併症

眼表面疾患は移植さえしてしまえば完全に「治す」ことができるのであろうか．否，世の中はそれほど甘くはない．角膜移植後に生じる治療困難な合併症には，移植片拒絶反応，免疫抑制に伴う感染症，続発緑内障などがある．なかでも続発緑内障は角膜移植後の失明原因として常に上位に位置しており，角膜がきれいになっても視神経が障害されてしまっては元も子もない．術後の一過性高眼圧を除いた角膜移植後続発緑内障の発症頻度は全体では30％程度とされているが，角膜移植の原因疾患によってその頻度が大きく異なっている．たとえば，無水晶体性水疱性角膜症では続発緑内障発症頻度が20～70％と最も高く，円錐角膜や先天性遺伝性角膜内皮ジストロフィ（CHED）などでは低いとされている[1]．

一般に，角膜移植後の続発緑内障発症・増悪のリスクとしては，角膜移植前から存在している緑内障の既往，無水晶体無硝子体眼，眼内レンズ（IOL）摘出術併用例などがあげられている．また眼圧上昇メカニズムとしては移植後の経過期間によりその機序が異なっており，移植後早期には前房内残留粘弾性物質，出血，炎症，角膜浮腫や浅前房による周辺虹彩前癒着，瞳孔ブロック，悪性緑内障などが考えられている．一方，移植後に一定の期間が経過した晩期では，ステロイド緑内障，拒絶反応に伴う炎症・虹彩前癒着の進行（図3），悪性緑内障などの機序があげられる．近年はシクロスポリンに代表されるステロイド以外の免疫抑制療法が広く用いられるようになり，ステロイドを長期にわたって使用しなくてはならない症例が減少してきているため，必然的にステロイド緑内障の頻度も減少傾向にある．いずれの時期においても角膜移植後に生じた続発緑内障は，角膜移植片の生着ならびに透明性維持にも悪影響を与え，かつ緑内障手術自体も炎症を惹起することから，移植片拒絶のリスクを高める結果となるため十分な管理上の注意が必要である．

II 角膜移植後緑内障の治療方針

治療の基本は他の続発緑内障と特に変わるところはない．視神経の障害程度を見きわめたうえで，必要最小限の薬物で最大限の効果を狙いつつ，眼圧上昇の原因に応じた治療を行う．しかし，角膜移植後眼の特殊性として基本的には眼底の透見性が不良なことが多いため，ついつい本来の対象臓器である前眼部ばかりに目が行きがちであり，気がついたら視野検査所見や視神経乳頭所見を取る努力すらなされていなかったという場合もある．眼圧測定も角膜表面が不整な状態であることが多い（図4）ため，トノペン®やアイケアなどの角膜接触面積の小さい眼圧計のほうが有利（図5）ではあるが，いかなる眼圧計を用いても正確に測定できず眼瞼上から触診で類推するしかない場合もある．いわんや接触検査である隅角検査などはもってのほかで，移植片拒絶反応や移植した上皮細胞の脱落の恐れから触ることすら制限されることがある．そのような場合でも緑内障手術適応決定時には隅角検査は必須であり，Susman型圧迫隅角鏡のような

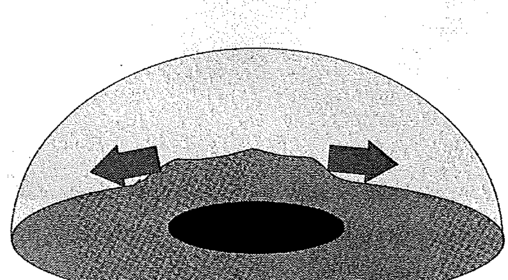

図3 Creepingメカニズムによる周辺虹彩前癒着の進行

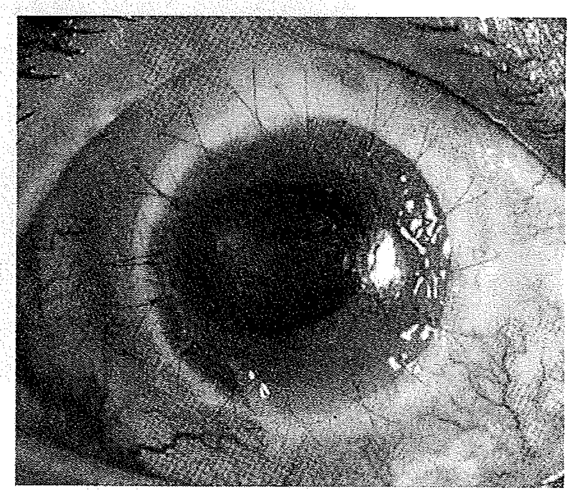

図4 全層角膜移植術後の眼表面不整

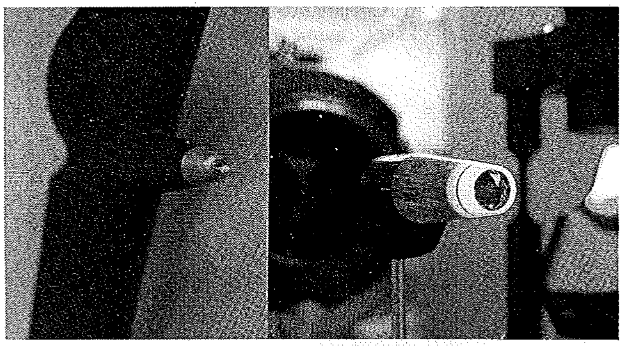

図5　アイケア眼圧計（左）とGoldmann眼圧計（右）の接触面積の差

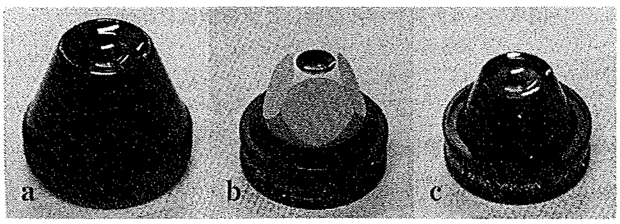

図6　各種隅角鏡の角膜接触面積の差
a：Goldmann三面鏡，b：Susman型圧迫隅角鏡，c：マグナビュー拡大一面鏡.

角膜接触面積の小さな隅角鏡（図6）を用いて，角膜上皮に負担をかけない隅角検査手技を日頃からマスターしておくことが必要である.

以上のように，緑内障治療の面からはかなり制限を課された状況下で診断・治療を進めてゆかねばならない点で，角膜移植後の緑内障は他の続発緑内障と比べてきわめて特殊であるといえる.

III　薬物療法の基本

通常，早期の眼圧上昇に対しては，βブロッカーやプロスタグランジン製剤などの点眼や炭酸脱水酵素阻害薬の点眼・内服治療から開始する．ステロイドは可能であるならば，より力価の弱いものに変更するか完全に中止するかして，シクロスポリンなどのステロイド以外の免疫抑制療法に切り替える．炭酸脱水酵素は角膜内皮にも存在し，角膜の透明性維持に関与していることから内皮細胞が減少しているような状況下では，可能な限り使用を制限したほうが望ましい．塩化ベンザルコニウムなどの防腐剤は眼表面に対して障害的に作用するため，可能な限り添加されていないかもしくは濃度が少ないほうが望ましい．しかし，抗緑内障薬の使用は眼圧下降を主目的としていることから，その決定にあたっては主剤の効果を基準にすべきであり，副作用を恐れて眼圧下降効果の劣る薬剤を投与することは本末転倒である．β_2交感神経刺激薬や副交感神経刺激薬は無水晶体眼における嚢胞様黄斑浮腫（CME）の危険性，虹彩前癒着の進行などの副作用を考慮したうえで，使用することのメリットがデメリットを上回るならば使用を考慮する．しかし，抗緑内障点眼薬はいずれも角膜の上皮障害ならびに結膜充血などの副作用をひき起こす可能性があるので，基本的には多剤併用は可能な限り3剤までにとどめるべきであり，点眼薬使用中には眼表面状態の十分な管理・注意が必要である.

IV　角膜移植後緑内障に対する手術療法

先に述べた内科的療法によっても眼圧がコントロールできない場合には，観血的療法を選択せざるをえない．選択的レーザー線維柱帯形成術などのレーザー治療は角膜の状態が良好でかつ開放隅角症例のみが適応となりうるが，角膜移植後緑内障における安全性と効果に関しては報告がない．角膜移植後緑内障に対する手術術式としては，移植した角膜内皮に対する影響や免疫抑制に伴う易感染性を考慮すれば，少しでも適応があるならまずはトラベクレクトミー（図7）よりもトラベクロトミーを選択したほうがよいと思われる（図8）．すなわち角膜の透明性が良好で隅角の状態が確認でき周辺虹彩前癒着が少ない症例では，眼圧上昇の機序としてステロイド緑内障の関与が強く疑われるため，トラベクロトミーが非常に良い適応となる．部分的な周辺虹彩前癒着が存在していても癒着期間が短い場合，もしくは面状ではなく線状に癒着をきたしている場合であれば隅角癒着解離術とトラベクロトミーの併用が可能である．しかし，実際には前述の条件を満たさないときには，他の難治続発緑内障の場合と同様にマイトマイシンC併用トラベクレクトミーが必要となる症例が多い．マイトマイシンC併用トラベクレクトミーの成功率は30～70％（経過観察期間1～5年）と報告によりさまざまであるが，角膜再移植例，無水晶体眼，周辺虹彩前癒着による閉塞隅角を認

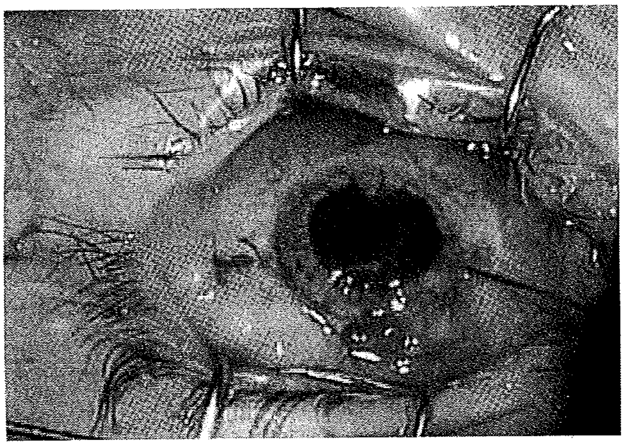

図7　全層角膜移植術後眼に対するマイトマイシンC併用線維柱帯切除術

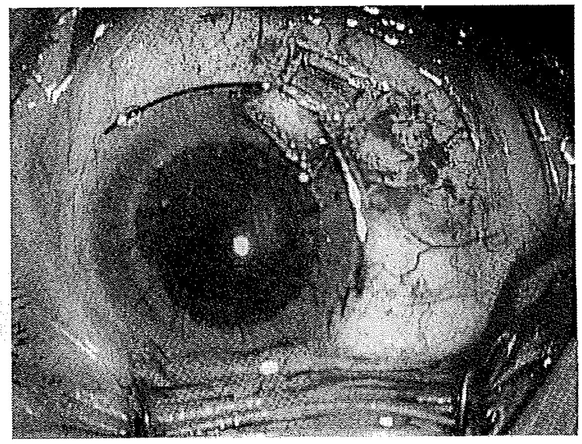

図8　全層角膜移植術後眼に対する線維柱帯切開術（下鼻側アプローチ）

める症例では眼圧コントロール成績が低下する[1]．また，5-フルオロウラシル（5-FU）の併用は角膜移植片における上皮障害と創傷治癒の遅延が懸念されるため，一般的には角膜移植後には禁忌とされる．さらに眼表面も同時に障害されている化学外傷やStevens-Johnson症候群，眼類天疱瘡など眼表面疾患を有する症例では，濾過胞作製部位の結膜も障害されていることからマイトマイシンC併用トラベクレクトミー自体が施行困難となるため，シャント手術の適応と考えられる．しかし，Molteno implantsやAhmed implantsなどのドレナージデバイス移植は，デバイス先端の角膜内皮への慢性的接触が発生するために，内皮脱落や移植片の拒絶反応などの合併症が多いのが難点である．毛様体扁平部から硝子体腔にデバイスの先を挿入する方法は角膜内皮への影響を最小化できるが，必ず硝子体手術を併用する必要があるため，角膜混濁による眼底透見性の程度が硝子体手術の可否を決める条件となる．眼内内視鏡を用いた硝子体手術も盛んに行われるようになっており，角膜混濁があっても硝子体手術の制限にはならなくなってきている．

以上のような観血的手術療法によっても眼圧コントロールが得られない症例には，最終手段としての毛様体破壊術を考慮せざるをえない．Nd-YAGレーザーや半導体レーザーを用いた経強膜毛様体レーザー光凝固術が最も一般的であるが，眼内炎症が遷延したような症例では毛様体突起の位置自体が大幅にずれている場合もあるために術後の眼圧予測がむずかしいだけでなく，惹起された炎症による移植片拒絶反応，低眼圧，視力低下，眼球癆などの合併症がある．

近年，これらの合併症を予防し過剰凝固を抑制するために，眼内内視鏡を用いて直視下に毛様体突起部のみを選択的に光凝固する手法や毛様体扁平部凝固など，種々の新しい手法が開発されてきている．

まとめ

角膜移植に伴う緑内障では，通常の緑内障検査を正確に行うことができないために，その診断が困難であり，発見や治療が遅れることが多い．しかし，緑内障の眼圧コントロールが角膜移植の治療成績にも影響することから，眼圧上昇時の抗緑内障点眼薬による角膜上皮障害には十分に注意して診察する必要がある．さらに眼表面炎症の抑制や角膜移植片の拒絶予防の目的で長期にわたりステロイドや免疫抑制薬を使用する例も多いため，ステロイド緑内障の発症のみならずトラベクレクトミー後の濾過胞感染にも十分な注意が必要である．

昨今，眼表面の再建方法が向上し，角膜移植や羊膜移植の適応疾患は広がりつつある．これに伴い眼表面全体に再建術を行っているような難症例に対する続発緑内障の治療も増加してきている．このような症例に対する治療として，現状のマイトマイシンC併用トラベクレクトミーでは限界がある．羊膜移植併用トラベクレクトミ

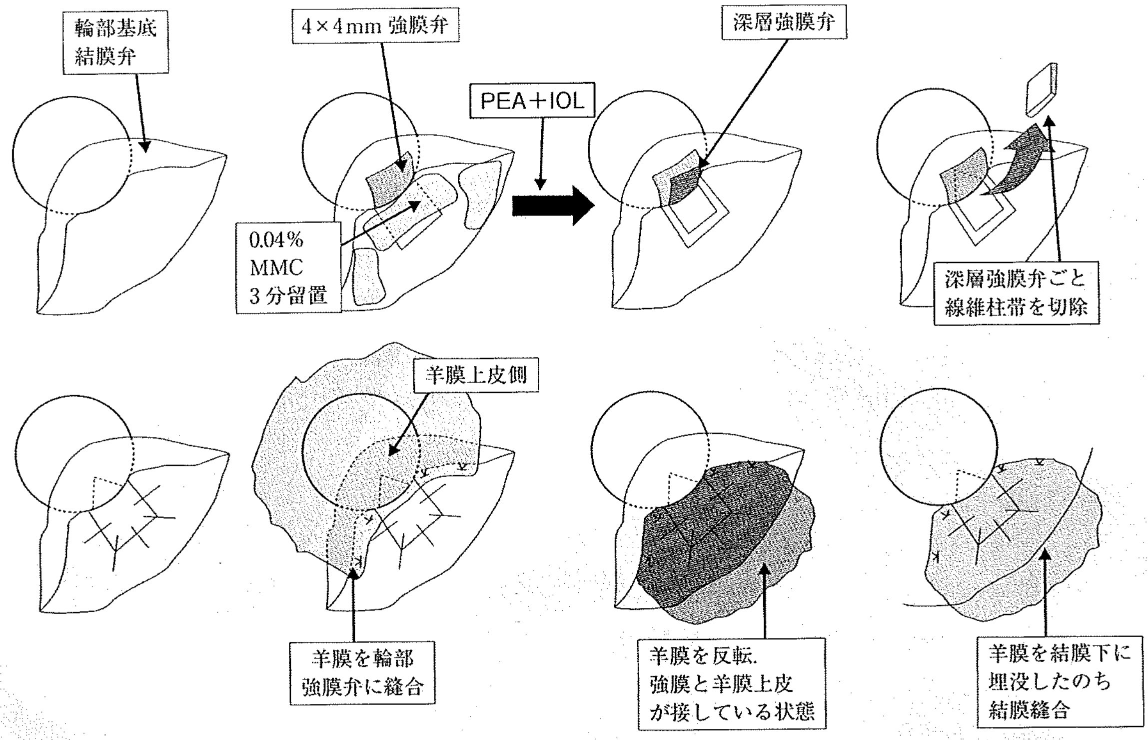

図9　羊膜移植併用トラベクレクトミー

ー（図9）は濾過胞の形成，強化に有効であるが，長期成績についてはまだ不明である[5]．今後は眼表面状態に左右されない新しい眼圧測定法や緑内障手術の開発が待ち望まれる．

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THE BORSUK-ULAM THEOREM IN A REAL CLOSED FIELD

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Abstract

Let C_p be a cyclic group of order p. We prove the definable C_p-version of the Borsuk-Ulam theorem in an o-minimal expansion $\mathcal{N} = (R, +, \cdot, <, \ldots)$ of a real closed field R.

1. Introduction

Any definable category is a generalization of the semialgebraic category. Many results in semialgebraic geometry hold true in the more general setting of an o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \ldots)$ of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. See also [5], [7], [16] for other examples and constructions of them. General references on o-minimal structures are [4], [6], [23], and there exist uncountably many o-minimal expansions of $\mathcal{R}$ [22].

Let $\mathcal{N} = (\mathbb{R}, +, \cdot, <, \ldots)$ be an o-minimal expansion of a real closed field R including $\mathbb{R}$. In this paper "definable" means "definable with parameters in $\mathcal{N}$", everything is considered in $\mathcal{N}$, each definable map is continuous and every simplicial complex K has the definably compact realization $|K|$ unless otherwise stated.

For a non-negative integer n, let $\mathbb{S}^n$ denote the n-dimensional unit sphere of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. Then it has the antipodal action of a cyclic group C_2 of order 2. The Borsuk-Ulam theorem states that if there exists a continuous C_2-map $f : \mathbb{S}^m \to \mathbb{S}^n$, then $m \leq n$ (e.g. P141 [15]). It is generalized to the case where a cyclic group C_p of order p acts on the odd dimensional sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ by $g(z_0, \ldots, z_n) = (e^{2\pi i/p} z_0, \ldots, e^{2\pi i/p} z_n)$, where g generates C_p ([13], [14]). There are several equivalent statements of it and many related generalizations (e.g. [2], [17], [18], [19], [20]).

In this paper, we prove the definable C_p-version of the Borsuk-Ulam theorem in $\mathcal{N}$.

For a non-negative integer n, the n-dimensional unit sphere S^n in R^{n+1} is defined by $S^n = \left\{(x_1, \ldots, x_{n+1}) \in R^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\right\}$. Then clearly S^n is a definable C^∞-manifold.

A *definable* $C^\infty C_p$*-manifold* is a pair (X, ϕ) consisting of a definable C^∞-manifold X and a group action $\phi : C_p \times X \to X$ which is a definable C^∞-map. We simply write X instead of (X, ϕ).

We consider the C_p-action on R^2 defined by

$$g\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 2\pi/p & -\sin 2\pi/p \\ \sin 2\pi/p & \cos 2\pi/p \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

where g generates C_p. Considering the diagonal action, we have a definable $C^\infty C_p$-action on R^{2m+2}. Then S^{2m+1} is C_p-invariant. Thus S^{2m+1} is a free definable $C^\infty C_p$-manifold.

Theorem 1.1 (Definable Borsuk-Ulam Theorem). (1) *Suppose that* $p \geq 3$ *and that* S^{2m+1} *and* S^{2n+1} *have the above* C_p*-actions. If there exists a definable* C_p*-map* $f : S^{2m+1} \to S^{2n+1}$, *then* $m \leq n$.

(2) *If* S^m *and* S^n *have the antipodal* C_2*-actions and there exists a definable* C_2*-map* $f : S^m \to S^n$, *then* $m \leq n$.

In Theorem 1.1(1), we cannot drop the condition that C_p acts freely on spheres. Even if $\mathcal{N} = \mathcal{R}$, there is a counterexample (see Proposition 2.10). Theorem 1.1 may be extensible to the case where spheres have general free definable C_p-actions.

2. Equivariant Definable Simplicial Complexes

By [4], every definable set X admits a finite partition into definable cells. Using cell decompositions, the Euler characteristic $\chi(X)$ of X is defined by the alternating sum of the number of i-dimensional cells. This number does not depend on the choice of definable cell decomposition and it is a definable bijective invariant (4.2.4 [4]).

On the other hand, the definable simplicial homology is introduced in [25]. For any definable set X, one can define the Euler characteristic $\chi(X)$ of X in a definable homological way. If X is definably compact, then these two definitions coincide [8]. If X is not definably compact, then they do not necessarily coincide. For example, if $X = [0, 1) \subset R$, then $\chi(X)$ in the first definition is 0 and $\chi(X)$ in the second definition is 1.

Let G be a finite group and $\Phi : G \to O_n(R)$ be a group homomorphism. A *representation* Ω of G means R^n with the orthogonal G-action induced by Φ. A G-invariant definable set X in a representation Ω of G is a *definable G-set* if X is G-invariant. In this paper, every representation is assumed to be orthogonal.

Proposition 2.1 (e.g. 10.2.18 [4]). *Let G be a finite group and X be a definable G-set. Then X/G is a definable subset of some R^n and the orbit map $p : X \to X/G$ is a surjective definably proper definable map.*

In $\mathcal{N}$, we can define equivariant simplicial complexes as follows when G is a finite group.

A *simplicial G-complex* consists of a simplicial complex K together with a G-action $\psi : G \times K \to K$ such that $\psi_g = \psi(g, \cdot) : K \to K$ is a simplicial homeomorphism for any $g \in G$.

Definition 2.2. A simplicial G-complex is an *equivariant simplicial complex* if the following two conditions are satisfied.

(1) For any subgroup H of G, if $\Delta = \langle v_0, \dots, v_n \rangle$ and $\Delta' = \langle h_0 v_0, \dots, h_n v_n \rangle$ are simplexes of K for $h_i \in H$, then there exists an $h \in H$ such that $hv_i = h_i v_i$ for all i.

(2) For every simplex Δ^n of K, the vertices $v_0, \dots, v_n$ of Δ^n can be ordered with $G_{v_n} \subset \cdots \subset G_{v_0}$.

Note that the second barycentric subdivision of any simplicial G-complex is an equivariant simplicial complex [10].

Definition 2.3. Let X be a definable G-set.

(1) An *equivariant definable triangulation* (L, ϕ) of X consists of an equivariant simplicial complex L and a definable G-homeomorphism ϕ from X to a G-invariant union of open simplexes of L.

(2) Let $X_1, \ldots, X_k$ be G-invariant definable subsets of X. An equivariant definable triangulation (L, ϕ) of X is *compatible with* $X_1, \ldots, X_k$ if each $\phi(X_i)$ is a G-invariant union of open simplexes of K.

Equivariant simplicial complexes and equivariant definable triangulations are introduced in [11] when $\mathcal{N} = \mathcal{M}$ and in [21] when $\mathcal{N} = \mathcal{R}$.

Let X be a definable G-set. A definable triangulation (L, ϕ) of the orbit space X/G is *compatible with the orbit types* if for any orbit type (H), $\phi \circ \pi(X(H))$ is a union of open simplexes of L, where $\pi : X \to X/G$ denotes the orbit map and $X(H) = \{x \in X \mid (G_x) = (H)\}$.

Theorem 2.4. *Let G be a finite group. Let X be a definable G-set in a representation Ω of G and $X_1, \ldots, X_k$ be G-invariant definable subsets of X. Then there exists an equivariant definable triangulation (K, ϕ) of X compatible with $X_1, \ldots, X_k$ and the orbit types. In particular, if X is definably compact, then $\phi(X) = |K|$.*

The following two results are the definable triangulation theorem (8.2.9 [4]) and piecewise triviality (9.1.7 [4]).

Theorem 2.5 (8.2.9 [4]). (*Definable triangulation theorem*) *Let X be a definable set and $X_1, \ldots, X_k$ be definable subsets of X. Then there exists a definable triangulation (M, τ) of X compatible with $X_1, \ldots, X_k$, namely M is a simplicial complex and τ is a definable homeomorphism from X to a union of open simplexes of M such that each $\tau(X_i)$ is a union of open simplexes of M. In particular, if X is definably compact, then $\tau(X) = |M|$.*

A definable map $f : X \to Y$ is *definably trivial* if there exist a point $y \in Y$ and a definable homeomorphism $H : X \to Y \times f^{-1}(y)$ such that $f = p \circ H$, where $p : Y \times f^{-1}(y) \to Y$ denotes the projection.

Theorem 2.6 (9.1.7 [4]). (*Piecewise triviality*) *Let X, Y be definable sets and $f : X \to Y$ be a definable map. Then there exists a finite partition $\{V_j\}_{j=1}^{u}$ of Y into definable sets such that each $f | f^{-1}(V_j)$ is definably trivial.*

Proof of Theorem 2.4. Let $(H_1), \ldots, (H_l)$ be orbit types of X.

First we prove the case where X is definably compact. Note that for each orbit type (H_i) of X, $X(H_i) = \{x \in X | G_x$ is conjugate to $H_i\}$ is G invariant. Since every $X(H_i)$ is a definable G-set and the orbit map $\pi : X \to X/G$ is a definable map, $\pi(X(H_i))$ is a definable subset of X/G. By Theorem 2.6, we have a finite partition $\{V_j\}_{j=1}^{u}$ of X/G such that each $\pi | \pi^{-1}(V_j)$ is definably trivial. Since X is definably compact and $\pi : X \to X/G$ is definable, X/G is definably compact. By Theorem 2.5, there exists a definable triangulation (M, τ) of X/G compatible with $\pi(X_1), \ldots, \pi(X_k)$, $\pi(X(H_1)), \ldots, \pi(X(H_l)), V_1, \ldots, V_u$.

Replacing M by its barycentric subdivision, if necessary, for each $\Delta \in M$, $\pi^{-1}(\tau^{-1}(\mathrm{Int}\Delta))$ is a definable G-set in Ω and there exists a definable section $s_\Delta : \tau^{-1}(\Delta) \to X$ such that:

(1) $s_\Delta(\tau^{-1}(\mathrm{Int}\Delta))$ is a definable set in Ω.

(2) $s_\Delta | \tau^{-1}(\mathrm{Int}\Delta) : \tau^{-1}(\mathrm{Int}\Delta) \to s_\Delta(\tau^{-1}(\mathrm{Int}\Delta))$ is a definable homeomorphism, where Int Δ denotes the interior of Δ.

Note that $s_\Delta(\tau^{-1}(\Delta)) = (\pi^{-1}(\tau^{-1}(\Delta)))^H$ and

$$s_\Delta(\tau^{-1}(\mathrm{Int}(\Delta))) = (\pi^{-1}(\tau^{-1}(\mathrm{Int}(\Delta))))^H,$$

where (H) is the orbit type of $\tau^{-1}(\mathrm{Int}\Delta)$.

Let L' be a finite abstract simplicial complex whose simplexes are $\{gs(\tau^{-1}(\Delta)) \mid \Delta \in K,\ g \in G\}$. Then L' is an abstract G-complex with the following G-action. For a simplex $\{hs(v_0), \ldots, hs(v_n)\}$ in L' and $g \in G$, we define $\phi_g(\{hs(v_0), \ldots, hs(v_n)\}) = \{ghs(v_0), \ldots, ghs(v_n)\}$. Now let L be the realization of L' and $\langle\langle gs(v_0), \ldots, gs(v_n)\rangle\rangle$ denote the corresponding simplex to $\{gs(\tau(v_0)), \ldots, gs(\tau(v_n))\}$ for $\Delta = \langle v_0, \ldots, v_n\rangle \in M$. Then L is an equivariant simplicial complex with the action induced by L'. For any $g \in G$, $\Delta = \langle v_0, \ldots, v_n\rangle$ in M and a section s on Δ, we define a linear homeomorphism $\prod_{\langle\langle gs(\Delta)\rangle\rangle} : \langle\langle gs(v_0), \ldots, gs(v_n)\rangle\rangle \to \Delta$ by $\prod_{\langle\langle gs(\Delta)\rangle\rangle}(gs(v_i)) = v_i$. Define $\prod = \cup \prod_{\langle\langle gs(\Delta)\rangle\rangle} : L \to M$. Then Π is a well-defined simplicial map. Note that $\prod : L \to M$ is the orbit map and $L/G = M$.

Define a map $\eta : |L| \to X$ by $\eta | \langle\langle gs(\Delta)\rangle\rangle = gs \circ \tau^{-1} \circ \prod | \langle\langle gs(\Delta)\rangle\rangle :$ $\langle\langle gs(\Delta)\rangle\rangle \to gs(\tau(\Delta))$, $\Delta \in M$, $g \in G$. Then η is a well-defined definable G-map. For each simplex $\Delta = \langle\langle gs(\Delta)\rangle\rangle$ of L, we have $\eta(\mathrm{Int}(\Delta)) = gs(\tau^{-1}(\mathrm{Int}(\Delta)))$. Moreover $\eta | \langle\langle gs(\mathrm{Int}(\Delta))\rangle\rangle$ is a definable G-homeomorphism from $\langle\langle gs(\mathrm{Int}(\Delta))\rangle\rangle$ onto its image because it is the composition of a linear isomorphism $\prod | \langle\langle gs(\mathrm{Int}\,\Delta)\rangle\rangle$, and definable homeomorphism $\tau^{-1} | \mathrm{Int}(\Delta)$ and gs_Δ. Therefore, (L, η^{-1}) is the required triangulation.

We now prove the general case. We can identify Ω with $\Omega \times \{1\} \subset \Omega \times R =: \Omega'$. Replacing Ω by Ω', we may assume that $0 \notin X$. Let $\theta : \Omega - \{0\} \to \Omega - \{0\}$ be $\theta(x) = \dfrac{x}{\| x \|^2}$. Then it is a Nash G-diffeomorphism because G acts on Ω orthogonally, where $\| x \|$ denotes the standard norm of Ω. Replacing X by $\theta(X)$, we may assume that X is bounded. Let $\overline{X}$ be the closure of X in Ω. Then $\overline{X}$ is a definably compact definable G-set in Ω. Apply $(\overline{X}, X, X_1, \ldots, X_k)$ to the definably compact case, we have an equivariant definable triangulation (K, ϕ') of $\overline{X}$ compatible with $X, X_1, \ldots, X_k$. Let $\phi = \phi' | X$. Then (K, ϕ) is the required triangulation. □

Using the proof of 5.22 [12] and Theorem 2.4, we obtain the following.

Theorem 2.7. *If a finite group G acts freely on a definably compact definable G-set X, then $\chi(X) = |G|\chi(X/G)$.*

By TR3. P786 [1], $H_0(S^n, \mathbb{Z}) \cong H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$ and for any i with $0 < i < n$, $H_i(S^n, \mathbb{Z}) \cong 0$. Thus $\chi(S^{2n}) = 2$. Therefore, a finite group whose order exceeds 3 cannot act freely on an even dimensional sphere. Thus it is justified to consider only odd-dimensional spheres in Theorem 1.1(1).

We now construct a counterexample of Theorem 1.1 when $\mathcal{N} = \mathcal{R}$ and G acts non-freely.

Let p, q be two distinct primes and $g = e^{\frac{2\pi}{pq}i}$. Then g generates a cyclic group C_{pq} of order pq. For any $k \in \mathbb{Z}$, let U_k be the complex plane $\mathbb{C}$ with the C_{pq}-action $g \cdot z = g^k z$ and U_k^t be the t-fold direct sum of U_k. Then for any $r, s \in \mathbb{Z}$, $t, u \in \mathbb{N} \cup \{0\}$, $U_r^t \oplus U_s^u$ is a representation of C_{pq}. Let $S(U_r^t \oplus U_s^u)$ denote the unit sphere of $U_r^t \oplus U_s^u$.

Theorem 2.8 [24]. *For any $t \in \mathbb{N}$, there exists a continuous C_{pq}-map $f : S(U_1^t) \to S(U_p \oplus U_q)$.*

Using the polynomial approximation theorem, existence of a Nash C_{pq}-tubular neighborhood of $S(U_p \oplus U_q)$ in $U_p \oplus U_q$ and averaging of the Haar measure, we have the following proposition.

Proposition 2.9. *Every continuous C_{pq}-map $f : S(U_1^t) \to S(U_p \oplus U_q)$ is G-homotopic to a definable $C^\infty C_{pq}$-map $h : S(U_1^t) \to S(U_p \oplus U_q)$.*

Note that C_{pq} acts freely on $S(U_1^t)$ and non-freely on $S(U_p \oplus U_q)$. By Proposition 2.9, we have the following proposition.

Proposition 2.10. *For any $t \in \mathbb{N}$, there exists a definable $C^\infty C_{pq}$-map $h : S(U_1^t) \to S(U_p \oplus U_q)$. In particular, there exists a definable C_{pq}-map from $S(U_1^t)$ to $S(U_p \oplus U_q)$.*

3. Proof of Theorem 1.1

Since $H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$, fix a generator x of $H_n(S^n, \mathbb{Z})$. For a definable map $f : S^n \to S^n$, it induces the homomorphism $(f_*)_n : H_n(S^n, \mathbb{Z}) \to H_n(S^n, \mathbb{Z})$. The degree of f is defined to be the integer $\deg f$ such that $(f_*)_n(x) = (\deg f)x$.

We prove the following theorem to prove Theorem 1.1.

Theorem 3.1. *Let G be a finite group acting freely on S^n. For every definable G-map $f : S^n \to S^n$, $\deg f \equiv 1 \bmod |G|$.*

Let $f : S^n \to S^n$ be a definable map. The Lefschetz number $L(f)$ of f is defined by $L(f) = \sum_{i=0}^{n}(-1)^i \operatorname{tr}(f_*)_i$, where $\operatorname{tr}(f_*)_i$ is the trace of the induced homomorphism $(f_*)_i : H_i(S^n, \mathbb{Z}) \to H_i(S^n, \mathbb{Z})$. The Lefschetz number $L(f)$ of f over $\mathbb{Q}$ is introduced in [8]. Since $H_0(S^n, \mathbb{Z}) \cong H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$ and for any i with $0 < i < n$, $H_i(S^n, \mathbb{Z}) \cong 0$, $L(f) = 1 + (-1)^n \deg f$. Thus we can easily obtain Theorem 3.1 from the following theorem.

Theorem 3.2. *Let G be a finite group acting freely on S^n. For every definable G-map $f : S^n \to S^n$, $L(f) \equiv 0 \bmod |G|$.*

For any equivariant simplicial complex X and for any $k \in \mathbb{N} \cup \{0\}$, let X_k denote the k-skeleton of X_k. A definable G-map $h : X \to Y$ between equivariant simplicial complexes is a *definable G-cellular map* if for each $k \in \mathbb{N} \cup \{0\}$, $h(X_k) \subset Y_k$.

Proposition 3.3 (4.1.6 [4]). *Let $h : X \to Y$ be a definable map between definable sets. Then* $\dim h(X) \leq \dim X$.

Under the assumption of Theorem 3.2, by Theorem 2.4, we may assume that S^n is an equivariant simplicial complex. Since f is a definable G-map, for each simplex σ, $f(\sigma)$ is a definable subset of S^n and $f(G\sigma)$ is a G-invariant definable subset of S^n. Thus using Theorem 2.4 and Proposition 3.3, we have the following lemma.

Lemma 3.4. *Let G be a finite group acting freely on S^n. For every definable G-map $h : S^n \to S^n$, after replacing some subdivision of S^n, we may assume that h is a definable G-cellular map $S^n \to S^n$.*

Let K be a simplicial complex of dimension n. For any i with $0 \le i \le n$, we define the i-dimensional chain complex $C_i(K)$ to be $H_i(K_i, K_{i-1})$ and the boundary homomorphism $\partial_i : C_i(K) \to C_{i-1}(K)$ to be the boundary homomorphism $\partial_* : H_i(K_i, K_{i-1}) \to H_{i-1}(K_{i-1}, K_{i-2})$ in the homology exact sequence. Then by Section V.6. [3], $\partial_i \circ \partial_{i+1} = 0$ and $C(X) = \{C_i(K), \partial_i\}$ is a free chain simplex. We call this the *chain complex* of the simplicial complex K. This chain complex defines the homology group of X.

By a way similar to P157 [9], we have the following.

Theorem 3.5. *Let X be a definably compact definable set. Then the definable homology group of X is isomorphic to the homology group defined from the chain complex of X.*

Proof of Theorem 3.2. By Lemma 3.4, we may assume that $f : S^n \to S^n$ is a definable G-cellular map. Thus f induces the homomorphism $f_* : C_i = H_i(S^n_i, S^n_{i-1}) \to C_i = H_i(S^n_i, S^n_{i-1})$. By Theorem 3.5, $L(f) = \sum_{i=0}^n (f_*)_i$. Since G acts freely on S^n, for any i with $0 \le i \le n$, $C_i(S^n)$ is a free $\mathbb{Z}G$-module, namely $C_i(S^n) = \mathbb{Z}g_1 e_{i1} \oplus \cdots \oplus \mathbb{Z}g_t e_{i1} \oplus \cdots \oplus \mathbb{Z}g_1 e_{il_i} \oplus \cdots \oplus \mathbb{Z}g_t e_{il_i}$, where $G = \{g_1, \ldots, g_t\}$ and $e_{i1}, \ldots, e_{il_i}$ denote i-dimensional simplexes. For any u, v, let $(f_*)_i(g_u e_{iv}) = \sum_{u,v} a_{i_u v} g_u e_{iv}$. Then since G acts freely on S^n, for any v, $a_{i_1 v} = \cdots = a_{i_t v}$. Thus the trace of $(f_*)_i$ is a multiple of $|G|$. Therefore, $L(f) \equiv 0 \bmod |G|$. □

Proof of Theorem 1.1. Suppose that $m > n$ and $i : S^{2n+1} \to S^{2m+1}$ is the inclusion. Since S^{2n+1} and S^{2m+1} have the actions as in introduction, i is a definable C_p-map. Thus $i \circ f : S^{2m+1} \to S^{2m+1}$ is a definable C_p-map. By Theorem 3.1, $\deg(i \circ f) \equiv 1 \bmod |G|$. In particular, $\deg(i \circ f) \neq 0$. On the other hand, $(i \circ f)_* = i_* \circ f_* = 0 : H_{2m+1}(S^{2m+1}, \mathbb{Z}) \to H_{2m+1}(S^{2m+1}, \mathbb{Z})$. It implies $\deg(i \circ f) = 0$. This contradiction proves Theorem 1.1. □

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A Note on the Existence Problem of Isovariant Maps between Representation Spaces

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A Note on the Existence Problem of Isovariant Maps between Representation Spaces

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Abstract. In this note, we discuss the converse of the isovariant Borsuk-Ulam theorem. In particular, we show that the converse holds for dihedral groups D_3, D_4 and D_6. On the other hand, the converse does not hold for other dihedral groups D_n, $n \neq 3, 4, 6$.

1. Introduction

Let G be a compact Lie group. A G-isovariant map $f : X \to Y$ between G-spaces is defined as a G-equivariant map preserving the isotropy subgroups, i.e, $G_x = G_{f(x)}$ for all $x \in X$. This notion was introduced by Palais [16] in order to study a classification problem for orbit maps of G-spaces.

In transformation group theory or equivariant topology, the existence and classification problem of G-maps is fundamental and important. Since Borsuk [2] proved the famous antipodal theorem, Borsuk-Ulam type theorems studied by many researchers, see for example [1], [4], [5], [6], [7]. These results are thought of as nonexistence results of G-maps, and have many applications; for example, Furuta [3] shows the 10/8-theorem in 4-dimensional topology using a Borsuk-Ulam type theorem, and Matoušek [8] illustrates several applications to combinatorics. Further results and applications on the Borsuk-Ulam theorem can be found in [14], [15].

Wasserman [18] first considered an isovariant version of the Borsuk-Ulam theorem. After that, Nagasaki [9], [10], [11] and Nagasaki-Ushitaki [13] also studied isovariant Borsuk-Ulam type theorems. One of Wasserman's results is the following.

Theorem 1.1 (Isovariant Borsuk-Ulam theorem). *Let G be a solvable compact Lie group and V, W (real) G-representation spaces; namely, G acts linearly on the (finite*

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