**Firm $y$'s global best response** 1. $V \leq \tau$

Note that $\widetilde{p}_y = V - \tau(1+y)$, for any $y \geq 0$, $\widetilde{p}_y \leq V - \tau \leq 0$. Therefore, the local best response for $p_y \leq \widetilde{p}_y$ is not relevant. We put together the local best responses for $\widetilde{p}_y \leq p_y \leq \underline{p}_y$ and $\underline{p}_y \leq p_y \leq p_y^0$ to get the global best response.

$$p_y = \begin{cases} \frac{(4V+x\tau+p_x)}{10} & \text{if } 6V - 10\tau - x\tau \leq p_x \leq \frac{16V-11x\tau}{11} \\ 2V - \tau x - p_x & \text{if } \frac{16V-11x\tau}{11} \leq p_x \leq \frac{3}{2}V - \tau x \\ \frac{V}{2} & \text{if } p_x \geq \frac{3}{2}V - \tau x \end{cases}.$$

and the locally optimal location is

$$y = 0.$$

$$\pi_y = \begin{cases} \frac{(4V+x\tau+p_x)^2}{40\tau} & \text{if } 6V - 10\tau - x\tau \leq p_x \leq \frac{16V-11x\tau}{11} \\ \frac{3(x\tau-V+p_x)(2V-x\tau-p_x)}{\tau} & \text{if } \frac{16V-11x\tau}{11} \leq p_x \leq \frac{3}{2}V - \tau x \\ \frac{3V^2}{4\tau} & \text{if } p_x \geq \frac{3}{2}V - \tau x \end{cases}.$$

2. $\tau \leq V \leq \frac{11}{5}\tau$

Putting different parts of the local best responses together gives

$$p_y = \begin{cases} \text{Not defined} & p_x \leq 3V - 7\tau - 5\tau x \\ \frac{3\tau+\tau x+V+p_x}{4} & \text{if } 3V - 7\tau - 5\tau x < p_x \leq 3V - 7\tau - x\tau \\ V - \tau & \text{if } 3V - 7\tau - x\tau \leq p_x \leq 6V - 10\tau - x\tau \\ \frac{(4V+x\tau+p_x)}{10} & \text{if } 6V - 10\tau - x\tau \leq p_x \leq \frac{16V-11x\tau}{11} \\ 2V - \tau x - p_x & \text{if } \frac{16V-11x\tau}{11} \leq p_x \leq \frac{3}{2}V - \tau x \\ \frac{V}{2} & \text{if } p_x \geq \frac{3}{2}V - \tau x \end{cases}.$$

and the locally optimal location is

$$y = \begin{cases} \text{Not defined} & p_x \leq 3V - 7\tau - 5\tau x \\ \frac{-7\tau+3V-x\tau-p_x}{4\tau} & \text{if } 3V - 7\tau - 5\tau x < p_x \leq 3V - 7\tau - x\tau \\ 0 & \text{if } p_x \geq 3V - 7\tau - x\tau \end{cases}.$$

$$\pi_y = \begin{cases} \text{Not defined} & p_x \leq 3V - 7\tau - 5\tau x \\ \frac{(V+3\tau+x\tau+p_x)^2}{16\tau} & \text{if } 3V - 7\tau - 5\tau x < p_x \leq 3V - 7\tau - x\tau \\ \frac{(5\tau-V+x\tau+p_x)(V-\tau)}{2\tau} & \text{if } 3V - 7\tau - x\tau \leq p_x \leq 6V - 10\tau - x\tau \\ \frac{(4V+x\tau+p_x)^2}{40\tau} & \text{if } 6V - 10\tau - x\tau \leq p_x \leq \frac{16V-11x\tau}{11} \\ \frac{3(x\tau-V+p_x)(2V-x\tau-p_x)}{\tau} & \text{if } \frac{16V-11x\tau}{11} \leq p_x \leq \frac{3}{2}V - \tau x \\ \frac{3V^2}{4\tau} & \text{if } p_x \geq \frac{3}{2}V - \tau x \end{cases}.$$

3. $\frac{11}{5}\tau \leq V \leq \frac{77}{17}\tau$ and $x \leq \frac{V-3\tau}{2\tau}$.

Putting different parts of local best responses together, we get

$$p_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq V - \tau(1+x) \\ V - \tau(1+x) & \text{if } V - \tau(1+x) < p_x \leq 3V - 7\tau - 5x\tau \\ \frac{3\tau + \tau x + V + p_x}{4} & \text{if } 3V - 7\tau - 5x\tau \leq p_x \leq 3V - 7\tau - x\tau \\ V - \tau & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

and the locally optimal location is

$$y = \begin{cases} \text{Not defined} & p_x \leq 3V - 7\tau - 5x\tau \\ \frac{-7\tau + 3V - x\tau - p_x}{4\tau} & \text{if } 3V - 7\tau - 5x\tau \leq p_x \leq 3V - 7\tau - x\tau \\ 0 & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

$$\pi_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq V - \tau(1+x) \\ 3(V - \tau(1+x)) & \text{if } V - \tau(1+x) < p_x \leq 3V - 7\tau - 5x\tau \\ \frac{(V + 3\tau + x\tau + p_x)^2}{16\tau} & \text{if} 3V - 7\tau - 5x\tau \leq p_x \leq 3V - 7\tau - x\tau \\ \frac{(5\tau - V + x\tau + p_x)(V - \tau)}{2\tau} & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

4. $\frac{11}{5}\tau \leq V \leq \frac{77}{17}\tau$ and $x \geq \frac{V-3\tau}{2\tau}$.

Putting different parts of local best responses together, we get

$$p_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq 3V - 7\tau - 5\tau x \\ \frac{3\tau + \tau x + V + p_x}{4} & \text{if } 3V - 7\tau - 5\tau x \leq p_x \leq 3V - 7\tau - x\tau \\ V - \tau & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

and the locally optimal location is

$$y = \begin{cases} \text{Not defined} & \text{if } p_x \leq 3V - 7\tau - 5\tau x \\ \frac{-7\tau + 3V - x\tau - p_x}{4\tau} & \text{if } 3V - 7\tau - 5\tau x \leq p_x \leq 3V - 7\tau - x\tau \\ 0 & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

$$\pi_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq 3V - 7\tau - 5\tau x \\ \frac{(V + 3\tau + x\tau + p_x)^2}{16\tau} & \text{if } 3V - 7\tau - 5\tau x \leq p_x \leq 3V - 7\tau - x\tau \\ \frac{(5\tau - V + x\tau + p_x)(V - \tau)}{2\tau} & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

5. $V \geq \frac{77}{17}\tau$ and $x \leq \frac{V-3\tau}{2\tau}$.

Putting different parts of local best responses together, we get

$$p_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq V - \tau(1+x) \\ V - \tau(1+x) & \text{if } V - \tau(1+x) < p_x \leq 3V - 7\tau - 5x\tau \\ \frac{3\tau + \tau x + V + p_x}{4} & \text{if } 3V - 7\tau - 5x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

and the locally optimal location is

$$y = \begin{cases} \text{Not defined} & p_x \leq 3V - 7\tau - 5x\tau \\ \frac{-7\tau+3V-x\tau-p_x}{4\tau} & \text{if } 3V - 7\tau - 5x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

$$\pi_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq V - \tau(1+x) \\ 3(V - \tau(1+x)) & \text{if } V - \tau(1+x) < p_x \leq 3V - 7\tau - 5x\tau \\ \frac{(V+3\tau+x\tau+p_x)^2}{16\tau} & \text{if} 3V - 7\tau - 5x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

6. $V \geq \frac{77}{17}\tau$ and $x \geq \frac{V-3\tau}{2\tau}$.

Putting different parts of local best responses together, we get

$$p_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq 3V - 7\tau - 5\tau x \\ \frac{3\tau+\tau x+V+p_x}{4} & \text{if } 3V - 7\tau - 5\tau x \leq p_x \leq V + \tau(1-x) \end{cases}.$$

and the locally optimal location is

$$y = \begin{cases} .\text{Not defined} & \text{if } p_x \leq 3V - 7\tau - 5\tau x \\ \frac{-7\tau+3V-x\tau-p_x}{4\tau} & \text{if } 3V - 7\tau - 5\tau x \leq p_x \leq V + \tau(1-x) \end{cases}.$$

$$\pi_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq 3V - 7\tau - 5\tau x \\ \frac{(V+3\tau+x\tau+p_x)^2}{16\tau} & \text{if } 3V - 7\tau - 5\tau x \leq p_x \leq V + \tau(1-x) \end{cases}.$$

## 4.2 The leader's optimal location and pricing

We discuss the first mover's optimisation problem in two cases: $x = 0$ and $x > 0$, taking into consideration of firm $y$'s best responses characterised in the previous section. Comparing these two cases gives us the global optimum for firm $x$.

### 4.2.1 If $x = 0$

Firm $y$'s best response is either $y > x$ with pricing behaviour given in Lemma 1 or $y = x = 0$ with pricing behaviour given in Lemma 2.

Define the price level which gives $\bar{t}_x = 1$ to be $\widetilde{p}_x$.

$$\frac{V - p_x}{\tau} = 1 \Longrightarrow \widetilde{p}_x = V - \tau.$$

Let $\underline{p}_x$ solves $\bar{t}_x = \underline{t}_y$. By definition, $\widetilde{p}_x < \underline{p}_x$.

For $p_x \leq \widetilde{p}_x$,

$$D_x = 2 + t_{xy} = 2 + \frac{p_y - p_x + \tau y}{2\tau}.$$

For $\widetilde{p}_x \leq p_x \leq \underline{p}_x$,

$$D_x = 2\frac{V - p_x}{\tau} + t_{xy} = \frac{4V + y\tau - 5p_x + p_y}{2\tau},$$

For $p_x \geq \underline{p}_x$,

$$D_x = 3\frac{V - p_x}{\tau}.$$

In this section, we take firm $x$'s location fix at $x = 0$ and analyse the optimal $p_x$ according to firm $y$'s best response. With $x = 0$, firm $y$ can choose $y > 0$ or $y = 0$. For every given $p_x$, firm $y$ can always choose to locate at $y = x = 0$ and get at least the same profit level as firm $x$ by undercutting $p_x$ marginally. Therefore, to make $(x = 0, y > 0)$ an equilibrium, firm $x$ needs to charge a price such that $\pi_y (y = x = 0) \leq \pi_y (x = 0, y > 0)$. Firm $x$'s optimal pricing is summarised in the following lemma with the proof collected in the appendix.

**Lemma 3** *For $x = 0$, firm $x$'s local optimal pricing and resulting equilibrium profit is*

$$p_x^* = \begin{cases} \frac{3-\sqrt{3}}{6}V & V \leq \frac{6}{9+\sqrt{3}}\tau \\ \frac{V+\tau-\sqrt{2(8V^2-9V\tau+3\tau^2)}}{5} & \frac{6}{9+\sqrt{3}}\tau \approx 0.56\tau \leq V \leq \frac{63-\sqrt{297}}{68}\tau \\ \frac{23V-\tau-\sqrt{48(2V-\tau)(5V+\tau)}}{49} & \frac{63-\sqrt{297}}{68}\tau \approx 0.673\tau \leq V \leq \left(6-\sqrt{24}\right)\tau \\ 23\tau - V - \sqrt{48\tau\left(11\tau - V\right)} & \left(6-\sqrt{24}\right)\tau \approx 1.1\tau \leq V \leq \frac{11-\sqrt{57}}{2}\tau \approx 1.73\tau \end{cases};$$

$$\pi_x = \begin{cases} \frac{V^2}{2\tau} \\ \frac{3\left(\sqrt{2(8V^2-9V\tau+3\tau^2)}+4V-\tau\right)\left(V-\sqrt{2(8V^2-9V\tau+3\tau^2)}+\tau\right)}{25\tau} \\ \frac{\left(9\sqrt{48(2V-\tau)(5V+\tau)}+136V+156\tau\right)\left(23V-\sqrt{48(2V-\tau)(5V+\tau)}-\tau\right)}{9604\tau} \\ \frac{\left(V-23\tau+4\sqrt{\tau(33\tau-3V)}\right)\left(3\tau-\sqrt{\tau(33\tau-3V)}\right)}{\tau} \end{cases}.$$

**Proof.** See the appendix. ■

### 4.2.2 If $x > 0$

With $x \geq y \geq 0$ and $r_x = r_y$, we only need to consider the case that $D_x$ comes from $r_x$. Define $\widetilde{p}_x$ to be the price level such that $\bar{t}_x = 1$. By definition

$$V - \tau(1-x) - \widetilde{p}_x = 0 \Rightarrow \widetilde{p}_x = V - \tau(1-x).$$

The optimal $p_x^* = \frac{V}{2}$ if $x$ can act as a local monoplist. Otherwise, $p_x = \widetilde{p}_x$. The following lemma presents $x$'s optimal pricing and location for the caes $x \geq y \geq 0$ and $r_x = r_y$.

**Lemma 4** *For $x > 0$, firm $x$'s optimal pricing and resulting equilibrium profit is*

$$(x, p_x) = \begin{cases} \left(x \in \left[\frac{V}{\tau}, \frac{2\tau - V}{2\tau}\right], \frac{V}{2}\right) & V \leq \frac{2}{3}\tau \\ \left(\frac{2\tau - V}{2\tau}, \frac{V}{2}\right) & \frac{2}{3}\tau \leq V \leq \frac{11}{16}\tau \\ \left(\frac{5V + 11\tau}{22\tau}, \frac{27V - 11\tau}{22}\right) & \frac{11}{16}\tau \leq V \leq \frac{319}{343}\tau \\ \left(\frac{47\tau - 23V}{36\tau}, \frac{13V + 11\tau}{36}\right) & \frac{319}{343}\tau \leq V \leq \frac{47}{23}\tau \end{cases}.$$

$$\pi_x = \begin{cases} \frac{V^2}{2\tau} & V \leq \frac{11}{16}\tau \\ \frac{(27V - 11\tau)(11\tau - 5V)}{242\tau} & \frac{11}{16}\tau \leq V \leq \frac{319}{343}\tau \\ \frac{(13V + 11\tau)^2}{1440\tau} & \frac{319}{343}\tau \leq V \leq \frac{47}{23}\tau \end{cases}.$$

**Proof.** See the appendix. ∎

### 4.2.3 Optimal $x$ and $p_x$

We now compare results in Lemmas 3 and 4 to get firm $x$'s optimal location and pricing. Our results show that firm $x$ always prefers to locate off the centre. The intuition is that if firm $x$ locates at the centre, given that firm $y$ can choose to locate at the centre and undercut $p_x$ marginally, the profit firm $x$ gets, in most cases, is equal to the profit of the second mover locating off the centre. However, if firm $x$ chooses to locate off the centre, since firm $y$ would prefer to locate at the centre, the profit level firm $x$ gets is the profit level for the first mover locating off the centre.

**Proposition 2** *For $V > \frac{6}{9+\sqrt{3}}\tau \approx 0.56\tau$, firm $x$ always prefers to locate off the centre with the optimal $(x, p_x)$ given in Lemma 4. For $V \leq \frac{6}{9+\sqrt{3}}\tau$, firm $x$ is indifferent between choosing $x = 0$ and $x > 0$ with $\pi_x = \frac{V^2}{2\tau}$ in both cases.*

**Proof.** See the appendix. ∎

## 5 Three Firm Analysis

For three firm oligopoly, we propose an equilibrium configuration and verify the parameter ranges to support it as an equilibrium. The equilibrium configuration we focus on is $x > 0$, $y > 0$, $z \geq 0$, $r_x \neq r_y \neq r_z$, and the three firms do not behave as local monopolists. With three firms, there is even less incentives for $x$ and $y$ to locate at the centre.

## 5.1 Firm $z$'s decision

We solve the game backwards starting with firm $z$'s decision. First, it is never optimal to have $\bar{t}_z > 1$. If this is the case, for any given price, $z$ would find it optimal to move towards the centre, since the demand coming from $r_z$ remains the same while the demand coming from the other two rays increase. Similarly, it is never optimal to have $\bar{t}_z < 1$. For any given prices, if $\bar{t}_z < 1$, $z$ can always prefer to move outwards. The demand would remain the same, while the firm would face less competition from the other two firms. Therefore, apart from the case where firms are local monopolists, in equilibrium, $(p_z, z)$ are chosen such that $\bar{t}_z = 1$. For $z \geq 0$, this is defined as

$$V - \tau\left(\bar{t}_z - z\right) - p_z = 0.$$

Imposing the condition $t_z = 1$ gives

$$z = \frac{p_z - V + \tau}{\tau}.$$

Let's focus on the equilibrium where (1) $D_z$ comes from $r_x$ and $r_y$ as well (2) firms are not local monopolists. In this case, $D_z = t_{xz} + t_{yz} + 1$. The marginal consumers are defined:

$$V - \tau\left(x - t_{xz}\right) - p_x = V - \tau\left(t_{xz} + z\right) - p_z.$$

This gives

$$t_{xz} = \frac{\tau\left(x - z\right) + p_x - p_z}{2\tau}.$$

Similarly,

$$t_{yz} = \frac{\tau\left(y - z\right) + p_y - p_z}{2\tau}.$$

With the constraint, $\bar{t}_z = 1$, the demand is

$$\begin{aligned} D_z &= \frac{\tau\left(x - z\right) + p_x - p_z}{2\tau} + \frac{\tau\left(y - z\right) + p_y - p_z}{2\tau} + 1 \\ &= \frac{\tau x + V - \tau + p_x - 2p_z}{2\tau} + \frac{\tau y + V - \tau + p_y - 2p_z}{2\tau} + 1. \end{aligned}$$

Firm $z$ solves the problem

$$\max_{p_z}\left(\frac{\tau x + V - \tau + p_x - 2p_z}{2\tau} + \frac{\tau y + V - \tau + p_y - 2p_z}{2\tau} + 1\right)p_z.$$

¿From FOC,

$$\left(\frac{-2}{2\tau} + \frac{-2}{2\tau}\right)p_z + \frac{\tau x + V - \tau + p_x - 2p_z}{2\tau} + \frac{\tau y + V - \tau + p_y - 2p_z}{2\tau} + 1 = 0,$$

we obtain,

$$p_z(p_x, x, p_y, y) = \frac{\tau x + 2V + p_x + \tau y + p_y}{8}, \qquad (3)$$

and

$$z(p_x, x, p_y, y) = \frac{\tau x + \tau y + p_x + p_y - 6V + 8\tau}{8\tau}. \qquad (4)$$

These are firm $z$'s best-response correspondences.[2]

## 5.2 Firm $y$'s decision

We analyse firm $y$'s decision given $(p_x, x)$ and anticipating reactions (3) and (4). First of all, the same argument in th previous section applies and in equilibrium, we have $\bar{t}_y = 1$, or

$$y = 1 - \frac{V - p_y}{\tau}.$$

The demand for firm $y$ is $1 - t_{yz}$ and profit is,

$$\pi_y = (1 - t_{yz})\, p_y = \left(1 - \frac{\tau\,(y - z) + p_y - p_z}{2\tau}\right) p_y$$

Substituting best-responses (3) and (4) and $\bar{t}_y = 1$,

$$\pi_y = \left(\frac{V + \tau + x\tau + p_x - 6p_y + 8\tau}{8\tau}\right) p_y.$$

The optimal price and locations are,

$$p_y(p_x, x) = \frac{V + 9\tau + \tau x + p_x}{12}, \quad y(p_x, x) = \frac{-11V + 21\tau + \tau x + p_x}{12\tau}. \qquad (5)$$

## 5.3 Firm $x$'s decision

Firm $x$'s demand is $1 - t_{xz}$, and profit is,

$$\pi_x = (1 - t_{xz})p_x = \left(1 - \frac{\tau\,(x - z) + p_x - p_z}{2\tau}\right) p_x.$$

Using the best-responses (3), (4) and (5) and restriction $\bar{t}_x = 1$, we have,

$$\pi_x = \frac{35\tau - 17p_x}{24\tau} p_x.$$

[2]The optimal choices for given $(p_x, x, p_y, y)$ are actually price given by (3) and any $0 \leq z \leq \frac{\tau x + \tau y + p_x + p_y - 6V + 8\tau}{8\tau}$.

The optimal price and locations are,

$$p_x^* = \frac{35}{34}\tau, \quad x^* = \frac{69}{34} - \frac{V}{\tau}.$$

¿From (3), (4) and (5), we have the optimal prices and location of the two other firms,

$$p_y^* = \frac{205}{204}\tau, \quad y^* = \frac{409}{204} - \frac{V}{\tau}, \quad p_z^* = \frac{619}{816}\tau, \quad z^* = \frac{1435}{816} - \frac{V}{\tau}.$$

The equilibrium marginal consumers are,

$$t_{xz}^* = \frac{13}{48}, \quad t_{yz}^* = \frac{67}{272}.$$

The equilibrium profits are,

$$\pi_x^* = \frac{1225}{1632}\tau < \pi_y^* = \frac{42025}{55488}\tau < \pi_z^* = \frac{383161}{332928}\tau.$$

## 5.4 Verification of the support for the equilibrium

For the above configuration to be supported as an equilibrium, the following conditions need to be satisfied.

1. Firms are not local monopolists.
2. With $r_z \neq r_y \neq r_x$, restrict the parameter ranges to the ones where $D_z$ comes from $r_x$ and $r_y$.
3. Make sure that $z$ does not want to choose to locate on $r_x$ and $r_y$.
4. Make sure that all natural restrictions are satisfied. That is, prices are non-negative and locations are within 0 and 1 (weekly, although in the porposed equilibrium, all locations should be strictly within 0 and 1). And it should be satisfied that $z < x = y$.

If these conditions are satisfied, the proposed equilibrium should indeed be an equilibirum for this three firm sequentical game.

(1) Exclude the parameter ranges where firms behave as local monopolists.

To do this, for $z > 0$, and $r_z \neq r_x \neq r_y$, we need to make sure that $\underline{t}_z \geq \underline{t}_x$ and $\underline{t}_z \geq \underline{t}_y$ at the equilibirum prices and locations. By definition

$$V - \tau\left(\underline{t}_z + z\right) - p_z = 0 \Rightarrow \underline{t}_z = \frac{V - p_z}{\tau} - z,$$

$$V-\tau\left(x-\underline{t}_{x}\right)-p_{x}=0 \Rightarrow \underline{t}_{x}=x-\frac{V-p_{x}}{\tau},$$

and

$$\underline{t}_{y}=y-\frac{V-p_{y}}{\tau}.$$

The conditions $\underline{t}_z \geq \underline{t}_x$ and $\underline{t}_z \geq \underline{t}_y$ are satisfied if

$$\frac{V-p_{z}}{\tau}-z \geq x-\frac{V-p_{x}}{\tau} \text{ and } \frac{V-p_{z}}{\tau}-z \geq y-\frac{V-p_{y}}{\tau}.$$

Substituting in the prices and locations from our proposed equilibrium gives

$$V \geq \frac{2275}{1632}\tau \approx 1.39\tau.$$

(2) With $r_z \neq r_y \neq r_x$, restrict the parameter ranges to the ones where $D_z$ comes from $r_x$ and $r_y$.

Here, the comparison should be made with the case $z>0$, $r_z \neq r_x \neq r_y$, and $D_z$ only comes from $r_z$. The cases where $r_z = r_x$ or $r_z = r_y$ will be dealt with separately. We argue that due to the sequence of the moves, if $D_x$ is restricted to cover only consumers on $r_z$, it should be the case that firms would choose $x > y \geq 0$ with $D_y$ coming from all three rays.

When we focus on the cases where firms are not local monopolists, the optimal price and location choice should satisfy the conditions that $\bar{t} = 1$ for all firms. We analyse the game backwards and start with firm $z$'s best response. As analysed previously, $\bar{t}_z = 1$ gives $z = \frac{p_z - V + \tau}{\tau}$. The marginal consumer $t_{yz}$ is determined by

$$V-\tau\left(z-t_{yz}\right)-p_{z}=V-\tau\left(y+t_{yz}\right)-p_{y}.$$

This gives$t_{yz} = \frac{\tau(z-y)+p_z-p_y}{2\tau}$. The demand for firm $z$ is therefore

$$D_{z}=1-t_{yz}=\frac{\tau+V+\tau y-2 p_{z}+p_{y}}{2 \tau}.$$

The optimisation problem is therefore

$$\max_{p_z}\left(\frac{\tau+V+\tau y-2 p_{z}+p_{y}}{2 \tau}\right) p_{z}.$$

The FOC gives

$$p_{z}=\frac{\tau+V+\tau y+p_{y}}{4} \text{ and } z=\frac{5 \tau-3 V+\tau y+p_{y}}{4 \tau}.$$

Now we turn to firm $y$'s decision. Firm $y$ is in competition with both firms. It takes $(p_x, x)$ as given and it takes into consideration firm $z$'s best response. We focus on the case where $D_y$ comes from all three rays. The condition $\bar{t}_y = 1$ gives $y = \frac{p_y - V + \tau}{\tau}$. The marginal consumer $t_{xy}$, located on $r_x$, is $\frac{\tau(x-y)+p_x-p_y}{2\tau}$. The demand for firm $y$ is therefore

$$D_y = 1 + \frac{\tau\,(x-y) + p_x - p_y}{2\tau} + \frac{\tau\,(z-y) + p_z - p_y}{2\tau}.$$

Substituting in the condition $\bar{t}_y = 1$ and $z$'s best response gives

$$D_y = 1 + \frac{\tau x + V + p_x - 3p_y}{2\tau}.$$

The optimisation problem for firm $y$ is

$$\max_{p_y} \left(1 + \frac{\tau x + V + p_x - 3p_y}{2\tau}\right) p_y.$$

The FOC and the condition $\bar{t}_y = 1$ give

$$p_y = \frac{2\tau + \tau x + V + p_x}{6} \text{ and } y = \frac{8\tau + \tau x + p_x - 5V}{6\tau}.$$

Finally, we analyse the choice of $(p_x, x)$ taking into consideration both firms $y$ and $z$'s best responses. The condition $\bar{t}_x = 1$ implies $x = \frac{p_x - V + \tau}{\tau}$. Substituting in $y$'s best responses gives

$$D_x = 1 - \frac{4p_x - 3\tau}{6\tau}.$$

The optimisation problem is

$$\max_{p_x} \left(1 - \frac{4p_x - 3\tau}{6\tau}\right) p_x.$$

The FOC and the condition $\bar{t}_x = 1$ give

$$p_x = \frac{9}{8}\tau \approx 1.125\tau \text{ and } x = \frac{17\tau - 8V}{8\tau}.$$

¿From the best responses and $\bar{t} = 1$, the prices and locations for the other two firms are

$$p_y = \frac{7\tau}{8} \approx 0.875\tau,\ y = \frac{15\tau - 8V}{8\tau},\ p_z = \frac{15\tau}{16} \approx 0.937\tau, \text{ and } z = \frac{31\tau - 16V}{16\tau}.$$

The marginal consumers are

$$t_{xy} = \frac{1}{4} \text{ and } t_{yz} = \frac{1}{16}.$$

The resulting profits for the firms are

$$\pi_x = (1 - t_{xy})\,p_x = \frac{27}{32}\tau \approx 0.84\tau,$$

$$\pi_y = (1 + t_{xy} + t_{yz})\,p_y = \frac{147\tau}{128} \approx 1.15\tau.$$

$$\pi_z = (1 - \ t_{yz})\,p_z = \frac{225}{256}\tau \approx 0.88\tau.$$

Firm $z$ gets higher profit when its demand comes from three rays. Furthermore, $\pi_z < \pi_y$. This can never be an equilibirum.

(3) Make sure that $z$ does not want to choose to locate on $r_x$ and $r_y$.

To achieve this, we impose the restriction that in the above proposed equilibrium, $z > 0$. Note that $D_z$ comes from all three rays in our proposed equilibrium. If in equilibrium, $z > 0$ and $r_z \neq r_x \neq r_y$, $z$ does not have incentive to move into rays $x$ and $y$. The condition we require is

$$\frac{1435}{816} - \frac{V}{\tau} > 0 \text{ or } V < \frac{1435}{816}\tau \approx 1.76\tau.$$

(4) All the equilibrium prices are positve, $p_i > 0$, and $\{x, y, z\} \in (0, 1)$.

The conditions are prices are satisfied. The restrictions on the positions give

$$0 < \frac{69}{34} - \frac{V}{\tau} < 1,$$

$$0 < \frac{409}{204} - \frac{V}{\tau} < 1,$$

and

$$0 < \frac{1435}{816} - \frac{V}{\tau} < 1.$$

To satisfiy the restions simultaneous, we have

$$\frac{35}{34}\tau \approx 1.03\tau < V < 1.76\tau.$$

Together with the restriction that firms are not local monopolists, we have the parameter range for our proposed equilibrium as

$$\frac{2275}{1632}\tau \approx 1.39\tau \leq V < 1.76\tau.$$

### 5.5 Comparison with duopoly equilibrium

¿From Lemma 4, we reproduce below the duopoly equilibrium for the parameter range $\frac{2275}{1632}\tau \approx 1.39\tau \leq V < 1.76\tau$. For firm $x$

$$x = \frac{47\tau - 23V}{36\tau},\ p_x = \frac{13V + 11\tau}{36},\ \text{and}\ \pi_x = \frac{(13V + 11\tau)^2}{1440\tau}.$$

For firm $y$, $y = 0$,

$$p_y = \frac{4V + x\tau + p_x}{10} = \frac{67V + 29\tau}{180},$$

and

$$\pi_y = \frac{(4V + x\tau + p_x)^2}{40\tau} = \frac{(67V + 29\tau)^2}{12\,960\tau}.$$

Compare three firm oligopoly and doupoly:

For the most specilised product: $p_z \geq p_y$ (Duopoly) if

$$\frac{619}{816}\tau \geq \frac{67V + 29\tau}{180}\ \text{or if}\ V \leq \frac{7313}{4556}\tau \approx 1.61\tau.$$

$p_x$ (Three firm) $\geq p_x$ (Duopoly) if

$$\frac{35}{34}\tau \geq \frac{13V + 11\tau}{36}\ \text{or if}\ V \leq \frac{443}{221}\tau \approx 2\tau.$$

$p_x$ (Three firm) $\geq p_y$ (Duopoly) if

$$\frac{35}{34}\tau \geq \frac{67V + 29\tau}{180}\ \text{or if}\ V \leq \frac{2657}{1139}\tau \approx 2.33\tau.$$

Therefore, for the parameter range we focus on, the prices for more specilised products always increase when the number of firms increase from 2 to 3. The price for the most standardised produce increases if $V$ is small and decreases if $V$ is large. Note that for the parameter range we focus on, the standard product is not offered in the three firm oligopoly.

## 6 Welfare Analysis

With product differentiation, the welfare effect of a price incrase upon entry is in general ambigous. Although we have demonstrated price increase with entry, given the new product in the market and less travel cost for the consumers, total welfare or even consumer surplus may still increase. We explore in this section the welfare comparison between duopoly and three firm oligopoly with the restriction on parameters, $\frac{2275}{1632}\tau \approx 1.39\tau \leq V < 1.76\tau$.

## 6.1 Duopoly

Note that for the relevant equilibrium, $x > 0$, $y = 0$, $V(t_{xy}) \geq 0$, $\bar{t}_x = 1$. The consumer surplus for consumers who purchase from $x$ is equal to

$$\begin{aligned} CS_x &= \frac{1}{2}(1-x)(V-p_x) + \frac{1}{2}(V - p_x + V(t_{xy}))(x - t_{xy}) \\ &= \frac{1}{2}(1-x)(V-p_x) + \frac{1}{2}(V - p_x + V - \tau(x - t_{xy}) - p_x)(x - t_{xy}). \end{aligned}$$

With $y = 0$,

$$t_{xy} = \frac{p_x - p_y + \tau x}{2\tau} = \frac{29\tau - 13V}{40\tau}.$$

This gives

$$\begin{aligned} CS_x &= \frac{1}{2}(1-x)(V-p_x) + \frac{1}{2}\left(2V - 2p_x - \tau\left(x - \frac{(29\tau - 13V)}{40\tau}\right)\right)\left(x - \frac{(29\tau - 13V)}{40\tau}\right) \\ &= \frac{1}{2}\left(1 - \frac{47\tau - 23V}{36\tau}\right)\left(V - \frac{13V + 11\tau}{36}\right) \\ &\quad + \frac{1}{2}\left(2V - 2\left(\frac{13V + 11\tau}{36}\right) - \tau\left(\frac{47\tau - 23V}{36\tau} - \frac{(29\tau - 13V)}{40\tau}\right)\right)\left(\frac{47\tau - 23V}{36\tau} - \frac{(29\tau - 13V}{40\tau}\right. \\ &= \frac{(23V - 11\tau)^2}{2592\tau} + \frac{(191V - 143\tau)(209\tau - 113V)}{86\,400\tau}. \end{aligned}$$

For consumers who purchase from $y$, the consumer surplus is equal to

$$CS_y = \bar{t}_y(V - p_y) + \frac{1}{2}(V - p_y + V(t_{xy}))(t_{xy}).$$

With $y = 0$,

$$\bar{t}_y = \frac{V - p_y}{\tau} = \frac{113V - 29\tau}{180\tau}.$$

This gives

$$\begin{aligned} CS_y &= \frac{113V - 29\tau}{180\tau}\left(V - \frac{67V + 29\tau}{180}\right) \\ &\quad + \frac{1}{2}\left(2V - 2\left(\frac{67V + 29\tau}{180}\right) - \tau\left(\frac{29\tau - 13V}{40\tau}\right)\right)\left(\frac{29\tau - 13V}{40\tau}\right) \\ &= \frac{(113V - 29\tau)^2}{32\,400\tau} + \frac{(569V - 377\tau)(29\tau - 13V)}{28\,800\tau}. \end{aligned}$$

## 6.2 Three firm oligopoly

For consumers purchasing from firm $x$

$$
\begin{aligned}
CS_x &= \frac{1}{2}(1-x)(V-p_x)+\frac{1}{2}(V-\tau(x-t_{xz})-p_x+V-p_x)(x-t_{xz}) \\
&= \frac{1}{2}\left(1-\left(\frac{69}{34}-\frac{V}{\tau}\right)\right)\left(V-\frac{35}{34}\tau\right) \\
&\quad +\frac{1}{2}\left(2V-\tau\left(\frac{69}{34}-\frac{V}{\tau}-\frac{13}{48}\right)-2\left(\frac{35}{34}\tau\right)\right)\left(\frac{69}{34}-\frac{V}{\tau}-\frac{13}{48}\right) \\
&= \frac{(34V-35\tau)^2}{2312\tau}+\frac{(2448V-3115\tau)(1435\tau-816V)}{1331\,712\tau}.
\end{aligned}
$$

For consumers who purchase from firm $y$

$$
\begin{aligned}
CS_y &= \frac{1}{2}(1-y)(V-p_y)+\frac{1}{2}(V-\tau(y-t_{yz})-p_y+V-p_y)(y-t_{yz}) \\
&= \frac{1}{2}\left(1-\left(\frac{409}{204}-\frac{V}{\tau}\right)\right)\left(V-\frac{205}{204}\tau\right) \\
&\quad +\frac{1}{2}\left(2V-\tau\left(\frac{409}{204}-\frac{V}{\tau}-\frac{67}{272}\right)-2\left(\frac{205}{204}\tau\right)\right)\left(\frac{409}{204}-\frac{V}{\tau}-\frac{67}{272}\right) \\
&= \frac{(204V-205\tau)^2}{83\,232\tau}+\frac{(816V-1025\tau)(1435\tau-816V)}{443\,904\tau}.
\end{aligned}
$$

For consumers who purchase from firm $z$

$$
\begin{aligned}
CS_z &= \frac{1}{2}(1-z)(V-p_z)+\frac{1}{2}(V-p_z+V-\tau(t_{xz}+z)-p_z)(z+t_{xz}) \\
&\quad +\frac{1}{2}(V-\tau z-p_z+V-\tau(z+t_{yz})-p_z)(t_{yz}) \\
&= \frac{1}{2}\left(1-\left(\frac{1435}{816}-\frac{V}{\tau}\right)\right)\left(V-\frac{619}{816}\tau\right) \\
&\quad +\frac{1}{2}\left(2V-2\left(\frac{619}{816}\tau\right)-\tau\left(\frac{13}{48}+\frac{1435}{816}-\frac{V}{\tau}\right)\right)\left(\frac{1435}{816}-\frac{V}{\tau}+\frac{13}{48}\right) \\
&\quad +\frac{1}{2}\left(2V-2\tau\left(\frac{1435}{816}-\frac{V}{\tau}\right)-2\left(\frac{619}{816}\tau\right)-\tau\left(\frac{67}{272}\right)\right)\left(\frac{67}{272}\right) \\
&= \frac{(816V-619\tau)^2}{1331\,712\tau}+\frac{(1224V-1447\tau)(69\tau-34V)}{27\,744\tau} \\
&\quad +\frac{67(3264V-4309\tau)}{443\,904}.
\end{aligned}
$$

The total consumer welfare is higher with three firm oligopoly if

$$\begin{aligned}
& \frac{(34V-35\tau)^2}{2312\tau}+\frac{(2448V-3115\tau)(1435\tau-816V)}{1331\,712\tau} \\
& +\frac{(204V-205\tau)^2}{83\,232\tau}+\frac{(816V-1025\tau)(1435\tau-816V)}{443\,904\tau} \\
& +\frac{(816V-619\tau)^2}{1331\,712\tau}+\frac{(1224V-1447\tau)(69\tau-34V)}{27\,744\tau} \\
& +\frac{67\,(3264V-4309\tau)}{443\,904} \\
\geq\ & \frac{(23V-11\tau)^2}{2592\tau}+\frac{(191V-143\tau)(209\tau-113V)}{86\,400\tau} \\
& +\frac{(113V-29\tau)^2}{32\,400\tau}+\frac{(569V-377\tau)(29\tau-13V)}{28\,800\tau}.
\end{aligned}$$

Or if

$$30\,877\,916V^2-105\,782\,296\tau V+89\,329\,469\tau^2\leq 0.$$

This holds for

$$\frac{1555\,622-\sqrt{33\,880\,360\,425}}{908\,174}\tau\approx 1.51\tau\leq V\leq\frac{1555\,622+\sqrt{33\,880\,360\,425}}{908\,174}\tau\approx 1.92\tau$$

Therefore, for our parameter range, total consumer welfare increases for $1.51\tau \leq V < 1.76\tau$.

## 7 Conclusion

We utilitise a new product space specification to study firms' incentive to customise. At first glance, the product space looks similar to Chen and Riordan (2005). However, the interpretation is very different. In Chen and Riordan, they do not analyse firms' location choice. Each firm provides one variety and is located at the end point of each spoke. In our model, the product space gives a natural interpretation of standard versus customised products. Our results indicate that in a sequential move game, the first mover always offers the standard product. The follower customises. Competition among firms softens as consumers' travelling costs increase.

## 8 Appendix

**Proof.** of Lemma 3

**Case (A) If** $p_x \leq V-\tau$

Firm $x$ solves

$$\max_{p_x} \left( \frac{4\tau + p_y - p_x + \tau y}{2\tau} \right) p_x = \frac{(11\tau - V - p_x)}{4\tau} p_x.$$

The FOC gives

$$p_x = \frac{11\tau - V}{2}.$$

Checking the boundary values:

$$\frac{11\tau - V}{2} \leq V - \tau \text{ if } V \geq \frac{13}{3}\tau \approx 4.33\tau.$$

For $y \geq 0$, we require

$$\frac{11\tau - V}{2} \geq 3V - 5\tau \text{ or } V \leq 3\tau.$$

Therefore, for $V \leq 3\tau$, $y > 0$ and the optimal $p_x^* = \widetilde{p}_x$. For $V \geq 3\tau$, the optimal $y = 0$ and $x = 0$ is never an equilibrium. At $p_x = V - \tau$,

$$\pi_y (y = x = 0) = 3 (V - \tau).$$

On the other hand, if $y$ locates at $y > 0$, $\pi_y = \frac{V^2}{4\tau}$. Firm $y$ has the incentive to locate at the centre if

$$\frac{V^2}{4\tau} \leq 3 (V - \tau).$$

Or if

$$V^2 - 12\tau V + 12\tau^2 \leq 0.$$

This holds for

$$\left(6 - \sqrt{24}\right) \tau \approx 1.1\tau \leq V \leq \left(6 + \sqrt{24}\right) \tau \approx 10.9\tau.$$

Therefore, for $\tau < V \leq \left(6 - \sqrt{24}\right) \tau \approx 1.1\tau$, firm $y$ does not have an incentive to locate at the centre and $p_x = V - \tau$ is a local maximum for $x = 0$. For $\left(6 - \sqrt{24}\right) \tau \leq V \leq 3\tau$, firm $y$ would have the incentive to locate at the centre and firm $x$ faces the price under-cutting constraint. To make firm $y$ indifferent between staying off the centre and locating at the centre, firm $x$ needs to charge a price such that

$$3p_x = \frac{(V + \tau + p_x)^2}{16\tau}.$$

This holds for

$$p_x = 23\tau - V \pm \sqrt{48\tau (11\tau - V)}.$$

Note that $23\tau - V - \sqrt{48\tau(11\tau - V)} > 0$.

$$23\tau - V - \sqrt{48\tau(11\tau - V)} \leq V - \tau$$

if $V^2 - 12V\tau + 12\tau^2 \leq 0$. This holds for the parameter range. Also $23\tau - V + \sqrt{48\tau(11\tau - V)} > V - \tau$.

Therefore, the local equilibrium in this case is that for $V \leq \left(6 - \sqrt{24}\right)\tau \approx 1.1\tau$, $p_x = V - \tau$ and for $\left(6 - \sqrt{24}\right)\tau \approx 1.1\tau \leq V \leq 3\tau$, $p_x = 23\tau - V - \sqrt{48\tau(11\tau - V)}$.

The constraint $y \geq 0$ requires

$$23\tau - V - \sqrt{48\tau(11\tau - V)} \geq 3V - 5\tau.$$

Or

$$(7\tau - V) \geq \sqrt{3\tau(11\tau - V)}$$

$$7\tau - V \geq 0 \text{ if } V \leq 7\tau.$$

Therefore, this does not hold for $V \geq 7\tau$. For $V < 7\tau$, this holds for

$$V \leq \frac{11 - \sqrt{57}}{2}\tau \approx 1.73\tau.$$

At $p_x = 23\tau - V - \sqrt{48\tau(11\tau - V)}$, the resulting

$$\pi_x = \frac{(11\tau - V - p_x)}{4\tau}p_x = \frac{\left(V - 23\tau + 4\sqrt{\tau(33\tau - 3V)}\right)\left(3\tau - \sqrt{\tau(33\tau - 3V)}\right)}{\tau}.$$

**Case (B) If** $V - \tau \leq p_x \leq \frac{5}{3}V - \tau$.

Firm $x$ solves

$$\max_{p_x} \pi_x = \frac{4V + y\tau - 5p_x + p_y}{2\tau}p_x = \frac{7Vp_x + 3\tau p_x - 9p_x^2}{4\tau}.$$

The FOC gives the local maximiser

$$p_x^* = \frac{7V + 3\tau}{18}.$$

Checking the boundary values:

$$\frac{7V + 3\tau}{18} \geq V - \tau \text{ if } V \leq \frac{21}{11}\tau \approx 1.91\tau.$$

$$\frac{7V + 3\tau}{18} \leq \frac{5}{3}V - \tau \text{ if } V \geq \frac{21}{23}\tau \approx 0.91\tau.$$

Therefore, for $V \leq \frac{21}{23}\tau$, the solution is a corner solution with the optimal $p_x^* = \frac{5}{3}V - \tau$. For $\frac{21}{23}\tau \leq V \leq \frac{21}{11}\tau$, the optimal $p_x = \frac{7V+3\tau}{18}$. For $\frac{21}{11}\tau \leq V$, the solution occurs at $p_x = V - \tau$. The analysis is performed in Case (A).

For $\frac{21}{23}\tau \approx 0.91\tau \leq V \leq \frac{21}{11}\tau$, $p_x = \frac{7V+3\tau}{18}$. If $y > 0$,

$$\pi_y = \frac{(V + \tau + p_x)^2}{16\tau} = \frac{(25V + 21\tau)^2}{5184\tau}.$$

¿From Lemma 2, if $y = x = 0$,

$$\pi_y \left( p_y = \frac{7V + 3\tau}{18} \right) = 3\frac{V - p_y}{\tau}(p_y) = \frac{(7V + 3\tau)(11V - 3\tau)}{108\tau}.$$

Firm $y$ would choose to locate at the centre and undercut $p_x$ if

$$\frac{(25V + 21\tau)^2}{5184\tau} \leq \frac{(7V + 3\tau)(11V - 3\tau)}{108\tau}.$$

This holds for

$$V \geq \frac{237 + \sqrt{2737\,152}}{3071}\tau \approx 0.62\tau.$$

Therefore, when the solution is interior, for the relevant parameter range, firm $y$ always has the incentive to locate at $y = x = 0$ and undercut $p_x$.To eliminate $y$'s incentive to undercut, $x$ needs to price such that

$$3\frac{V - p_x}{\tau}(p_x) = \frac{(V + \tau + p_x)^2}{16\tau}.$$

This gives the constrained price

$$p_x = \frac{23V - \tau - \sqrt{48(2V - \tau)(5V + \tau)}}{49}.$$

Note that $\left.\frac{\partial(3\frac{V-p_x}{\tau}p_x)}{\partial p_x}\right|_{p_x = \frac{23V - \tau - \sqrt{48(2V-\tau)(5V+\tau)}}{49}} > 0$ and indeed firm $y$ would not have the incentive to undercut $p_x$ at this range..This solution falls into the relevant parameter range if

$$\frac{23V - \tau - \sqrt{48(2V - \tau)(5V + \tau)}}{49} \leq \frac{5}{3}V - \tau.$$

This is true for the relevant parameter range.

$$\frac{23V - \tau - \sqrt{48(2V - \tau)(5V + \tau)}}{49} \geq V - \tau$$

if

$$24\tau - 13V \geq \sqrt{12(2V - \tau)(5V + \tau)}.$$

$$24\tau - 13V \geq 0 \text{ if } V \leq \frac{24}{13}\tau \approx 1.85\tau.$$

Therefore, the inequality does not hold for $V \geq \frac{24}{13}\tau \approx 1.85\tau$. For $V \leq \frac{24}{13}\tau \approx 1.85\tau$, the inequality holds if

$$V \leq \left(6 - \sqrt{24}\right)\tau \approx 1.1\tau.$$

Therefore, for $\frac{21}{23}\tau \leq V \leq \left(6 - \sqrt{24}\right)\tau$, the local optimal price for firm $x$ is $p_x = \frac{23V - \tau - \sqrt{48(2V-\tau)(5V+\tau)}}{49}$. The resulting profit for firm $x$ is

$$\Pi_x = \frac{\left(9\sqrt{48(2V-\tau)(5V+\tau)} + 136V + 156\tau\right)\left(23V - \sqrt{48(2V-\tau)(5V+\tau)} - \tau\right)}{9604\tau}.$$

**Case (C) If $\frac{5}{3}V - \tau < p_x < 2V - \tau$**

In this case, given $p_x$, if $y > 0$, firm $y$ chooses $(y, p_y)$ such that $\bar{t}_x = \underline{t}_y$. If $y = 0$, $y$ chooses $p_y = p_x - \varepsilon$. In equilibrium $\varepsilon \to 0$.

$$\frac{5}{3}V - \tau \leq \frac{V}{2} \text{ if } V \leq \frac{6}{7}\tau \approx 0.86\tau.$$

For $\frac{6}{7}\tau \leq V \leq \frac{21}{23}\tau$: both local maximisers fall outside of the relevant regions and the global maximum occurs at $p_x^* = \frac{5}{3}V - \tau$ with the resulting profit

$$\pi_x = 3\frac{V - p_x}{\tau}p_x = \frac{(5V - 3\tau)(3\tau - 2V)}{3\tau}.$$

For $p_x = \frac{5}{3}V - \tau$ and $y > 0$,

$$\pi_y = \frac{(\tau - V + p_x)(3V - \tau - p_x)}{2\tau} = \frac{4V^2}{9\tau}.$$

Firm $y$ would have the incentive to locate at the centre for the relevant parameter range. Therefore, firm $x$ needs to price below $p_x = \frac{5}{3}V - \tau$ and eliminate firm $y$'s incentive to locate at the centre. For $p_x \leq \frac{5}{3}V - \tau$, $p_y$'s best response is analysed in Case (B) above.

For $\frac{2}{3}\tau \leq V \leq \frac{6}{7}\tau$, $\frac{5}{3}V - \tau \leq \frac{V}{2} \leq 2V - \tau$ and the local optimal $p_x$ without the price undercutting constraint is $p_x = \frac{V}{2}$. Firm $y$'s best response when $y > 0$ is to price such that $\underline{t}_y = \bar{t}_x$. Since firm $x$ gets effectively the unconstrained monopoly profit, firm $y$ would always get less profit locating off the centre and would always has the incentive to locate at $y = 0$. To eliminate firm $y$'s incentive to undercut, firm $x$ needs to price such that

$$3\frac{V - p_x}{\tau}p_x = \frac{(\tau - V + p_x)(3V - \tau - p_x)}{2\tau}.$$

This gives the constrained $p_x$

$$p_x = \frac{V + \tau - \sqrt{2\left(8V^2 - 9V\tau + 3\tau^2\right)}}{5}.$$

Note that $\frac{V+\tau-\sqrt{2(8V^2-9V\tau+3\tau^2)}}{5} \geq 0$ in the relevant parameter range.

$$\frac{V + \tau - \sqrt{2\left(8V^2 - 9V\tau + 3\tau^2\right)}}{5} \geq \frac{5}{3}V - \tau$$

if

$$6\tau - \frac{22}{3}V \geq \sqrt{2\left(8V^2 - 9V\tau + 3\tau^2\right)}.$$

This does not hold if

$$6\tau - \frac{22}{3}V \leq 0 \text{ or } V \geq \frac{9}{11}\tau \approx 0.82\tau.$$

For $V \leq \frac{9}{11}\tau \approx 0.82\tau$, this holds for

$$V \leq \frac{63 - \sqrt{297}}{68}\tau \approx 0.673\tau.$$

$$\frac{V + \tau - \sqrt{2\left(8V^2 - 9V\tau + 3\tau^2\right)}}{5} \leq 2V - \tau$$

if

$$6\tau - 9V \leq \sqrt{2\left(8V^2 - 9V\tau + 3\tau^2\right)}.$$

This holds in the relevant parameter range.

Therefore, for $\frac{63-\sqrt{297}}{68}\tau \approx 0.67\tau \leq V \leq \frac{6}{7}\tau$, the equilibirum falls into the first part of firm $y$'s best response. The analysis is presented in Case (B). For $\frac{2}{3}\tau \leq \tau \leq \frac{63-\sqrt{297}}{68}\tau \approx 0.67\tau$, the optimal $p_x$ is $p_x = \frac{V+\tau-\sqrt{2(8V^2-9V\tau+3\tau^2)}}{5}$. The resulting $\pi_x$ is

$$\pi_x = \frac{3\left(\sqrt{2\left(8V^2 - 9V\tau + 3\tau^2\right)} + 4V - \tau\right)\left(V - \sqrt{2\left(8V^2 - 9V\tau + 3\tau^2\right)} + \tau\right)}{25\tau}.$$

**Case (D) If** $2V - \tau \leq p_x$

When $p_x \geq 2V - \tau$, both firms act as local monopolist. The relevant demand is $D^x = 3\frac{V-p_x}{\tau}$. The local maximiser is interior if

$$\frac{V}{2} \geq 2V - \tau \text{ or } V \leq \frac{2}{3}\tau.$$