Proof. Since $C_{k-1}(M)$ is a free $\mathbb{Z}C_n$-module, it follows from [3, II, (4.5)] that $\bar{\tau}: C^{k-1}(M) \to C^{k-1}_{C_n}(M)$ is surjective, and hence so is τ.

Since M is orientable, it holds that

$$H^{k-1}\big(M; \pi_{*-1}(SW_{\text{free}})\big) \cong \pi_{k-1}(SW_{\text{free}}) \cong \bigoplus_{H \in \mathcal{A}} \mathbb{Z},$$

and

$$\mathfrak{H}^{k-1}_{C_n}\big(M; \pi_{k-1}(SW_{\text{free}})\big) \cong \bigoplus_{H \in \mathcal{A}} \mathbb{Z}$$

as in the proof of Lemma 6.3. Hence $\mathfrak{H}^{k-1}_{C_n}(M; \pi_{k-1}(SW_{\text{free}}))$ and $H^{k-1}(M; \pi_{k-1}(SW_{\text{free}}))$ are free Abelian groups with the same rank. This leads that τ is an isomorphism. □

Here we give a cohomological description of the multidegree. Let $f: M \to SW_{\text{free}}$ be a C_n-map. Using the universal coefficient theorem, we have the following isomorphisms:

$$\begin{aligned} H^{k-1}\big(SW_{\text{free}}; \pi_{k-1}(SW_{\text{free}})\big) &\xrightarrow[\cong]{\kappa} \operatorname{Hom}_{\mathbb{Z}}\big(H_{k-1}(SW_{\text{free}}), \pi_{k-1}(SW_{\text{free}})\big) \\ &\xrightarrow[\cong]{h^*} \operatorname{Hom}_{\mathbb{Z}}\big(\pi_{k-1}(SW_{\text{free}}), \pi_{k-1}(SW_{\text{free}})\big), \end{aligned}$$

where h denotes the Hurewicz homomorphism. Set

$$\iota(SW_{\text{free}}) = (h^* \circ \kappa)^{-1}(id_{\pi_{k-1}(SW_{\text{free}})}) \in H^{k-1}\big(SW_{\text{free}}; \pi_{k-1}(SW_{\text{free}})\big).$$

Then one can see that

$$\langle f^*\iota(SW_{\text{free}}), [M]\rangle = f_*([M]) \in \pi_{k-1}(SW_{\text{free}}),$$

where $f_* \in \operatorname{Hom}(H_{k-1}(M; \mathbb{Z}), \pi_{k-1}(SW_{\text{free}}))$ and $[M]$ is the fundamental class of M. We obtain that

$$\operatorname{mDeg} f = \Phi \circ h\big(\langle f^*\iota(SW_{\text{free}}), [M]\rangle\big).$$

Proof of Theorem D. (1) It is well known [3, II, (3.19)] that

$$\gamma(f, g) = f^*\iota(SW_{\text{free}}) - g^*\iota(SW_{\text{free}}) \in H^{k-1}\big(M, \pi_{k-1}(SW_{\text{free}})\big).$$

Hence we obtain that

$$\operatorname{mDeg} f - \operatorname{mDeg} g = \Phi \circ h\big(\langle \gamma(f, g), [M]\rangle\big).$$

Assume $\operatorname{mDeg} f = \operatorname{mDeg} g$, then it holds that $\gamma(f, g) = 0$. Since $\gamma(f, g) = \varepsilon(\gamma_{C_n}(f, g))$ and ε is injective, we have $\gamma_{C_n}(f, g) = 0$. Thus f and g are C_n-homotopic.

(2) By Lemmas 6.2 and 6.4, we have $\operatorname{Image}(\varepsilon) = nH^{k-1}(M; \pi_{k-1}(SW_{\text{free}}))$. Thus we obtain $\gamma(f, g) = \varepsilon(\gamma_{C_n}(f, g)) \in nH^{k-1}(M; \pi_{k-1}(SW_{\text{free}}))$ and hence it holds that

$$\operatorname{mDeg} f - \operatorname{mDeg} g = \Phi \circ h\big(\langle \varepsilon(\gamma_{C_n}(f, g)), [M]\rangle\big) \in \bigoplus_{H \in \mathcal{A}} n\mathbb{Z}.$$

(3) Note that

$$\operatorname{Image}(\varepsilon) = nH^{k-1}\big(M; \pi_{k-1}(SW_{\text{free}})\big) \cong \bigoplus_{H \in \mathcal{A}} n\mathbb{Z}.$$

Every element of $\mathfrak{H}^{k-1}_{C_n}(M; \pi_{k-1}(SW_{\text{free}}))$ is realized as $\gamma_{C_n}(f, f_0)$ for some C_n-map $f: M \to SW_{\text{free}}$. Therefore, any element of $\bigoplus_{H \in \mathcal{A}} n\mathbb{Z}$ is realized by $\operatorname{mDeg} f - \operatorname{mDeg} f_0$. □

7. Examples

Here we give a couple of examples. Put $n = pq$, where p and q are distinct primes.

Let g be a generator of C_n. Let T_m denote the irreducible unitary representation of C_n given by $\rho_m : C_n \to U(1)$ defined by $\rho_m(g)(z) = \zeta^m z$, where $\zeta = \exp(2\pi i/n)$. When we denote the greatest common divisor of two integers m and n by (m, n), we note that $\ker \rho_m = C_{(m,n)}$.

Example 7.1. Suppose $M = ST_1$ and $SW = S(T_p \oplus T_q)$. We determine the structure of $[M, SW]_{C_n}^{\mathrm{isov}}$. Since $SW^{C_p} = S(T_p)$ and $SW^{C_q} = S(T_q)$, we have $SW^{>1} = S(T_p) \sqcup S(T_q)$, which is a Hopf-link in SW. Hence, it holds that $\dim M + 1 = k = 2$, which shows the existence of C_n-isovariant maps from M to SW.

Next, we classify the C_n-isovariant maps. We see that $\mathcal{A} = \{C_p, C_q\}$. Now we define a map $f_{\alpha,\beta} : ST_1 \to S(T_p \oplus T_q)$ by

$$f_{\alpha,\beta}(z) = \frac{1}{\sqrt{2}}\big(z^{(1+\alpha q)p}, z^{(1+\beta p)q}\big),$$

where $\alpha, \beta \in \mathbb{Z}$. Then, one can easily check that it is a C_n-isovariant map. By the definition of Φ, we obtain

$$\mathrm{mDeg}\, f_{\alpha,\beta} = \big(\deg\big(z \mapsto z^{(1+\beta p)q}\big), \deg\big(z \mapsto z^{(1+\alpha q)p}\big)\big) = \big((1+\beta p)q, (1+\alpha q)p\big).$$

If taking $f_{0,0}$ as the reference map, we can construct a bijection

$$\mathrm{mD}_{f_{0,0}} : \big[ST_1, S(T_p \oplus T_q)\big]_{C_n}^{\mathrm{isov}} \to \mathbb{Z} \oplus \mathbb{Z}$$

by

$$\mathrm{mD}_{f_{0,0}}(f_{\alpha,\beta}) = (\mathrm{mDeg}\, f_{\alpha,\beta} - \mathrm{mDeg}\, f_{0,0})/n = (\beta, \alpha).$$

Remark 7.2. In the setting of Example 7.1, all $f_{\alpha,\beta}$ are C_n-homotopic each other.

Example 7.3. Here, we consider more general example than Example 7.1, that is, put $M = ST_1$ and $SW = S(rT_1 \oplus sT_p \oplus tT_q)$ $(t \geqq s)$, where rT_1 means the direct sum of r-copies of T_1, and so on. If $\dim M + 1 \leqq k$, it holds that $k \geqq 2$. Since the representation is assumed to be faithful, one can easily check that $r \neq 0$ or $s \neq 0$.

Since $SW^{C_p} = S(sT_p)$ and $SW^{C_q} = S(tT_q)$, we see that $SW^{>1} = S(sT_p) \sqcup S(tT_q)$ and then $k = 2(r+s) \geqq 2$, which shows the existence of C_n-isovariant maps from M to SW. If $k > 2$, all C_n-isovariant maps from M to SW are C_n-isovariantly homotopic each other. If $k = 2$, we have $s = 0$, $r = 1$ or $s = 1$, $r = 0$, and use the multidegree for our classifying problem.

Case I *($s = 0$, $r = 1$).* In this case, $SW = S(T_1 \oplus tT_q)$ and $SW^{>1} = S(tT_q)$. If $t = 0$, the action is free. If $t \neq 0$, we have $\mathrm{Iso}(SW) = \{1, C_q\}$ and $\mathcal{A} = \{C_q\}$. Now we define a map $f_\alpha : ST_1 \to S(T_1 \oplus tT_q)$ by

$$f_\alpha(z) = \big(z^{1+\alpha n}, *, \ldots, *\big)/\|-\|,$$

where $\alpha \in \mathbb{Z}$. Then, one can easily verify that it is a C_n-isovariant map. By the definition of Φ, we obtain

$$\mathrm{mDeg}\, f_\alpha = \deg\big(z \mapsto z^{1+\alpha n}\big) = 1 + \alpha n.$$

If choosing f_0 as the reference map, we can construct a bijection $\mathrm{mD}_{f_0} : [ST_1, S(T_1 \oplus tT_q)]_{C_n}^{\mathrm{isov}} \cong \mathbb{Z}$ by

$$\mathrm{mD}_{f_0}\big([f_\alpha]\big) = (\mathrm{mDeg}\, f_\alpha - \mathrm{mDeg}\, f_0)/n = \alpha.$$

Case II *($s = 1$, $r = 0$).* In this case, $SW = S(T_p \oplus tT_q)$, $SW^{>1} = ST_p \sqcup S(tT_q)$ and $\mathrm{Iso}(SW) = \{1, C_p, C_q\}$. If $t = 1$, the problem was already discussed in Example 7.1. If $t \geqq 2$, we have $\mathcal{A} = \{C_q\}$. Now we define a map

$$f_\alpha : ST_1 \to S(T_p \oplus tT_q)$$

by

$$f_\alpha(z) = \big(z^{(1+\alpha q)p}, *, \ldots, *\big)/\|-\|,$$

where $\alpha \in \mathbb{Z}$. Then, one can easily check that it is a C_n-isovariant map. By the definition of Φ, we obtain

$$\mathrm{mDeg}\, f_\alpha = \deg\bigl(z \mapsto z^{(1+\alpha q)p}\bigr) = (1 + \alpha q)p.$$

If taking f_0 as the reference map, we can construct a bijection

$$\mathrm{mD}_{f_0} : \bigl[ST_1, S(T_p \oplus tT_q)\bigr]^{\mathrm{isov}}_{C_n} \to \mathbb{Z}$$

by

$$\mathrm{mD}_{f_0}\bigl([f_\alpha]\bigr) = (\mathrm{mDeg}\, f_\alpha - \mathrm{mDeg}\, f_0)/n = \alpha.$$

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