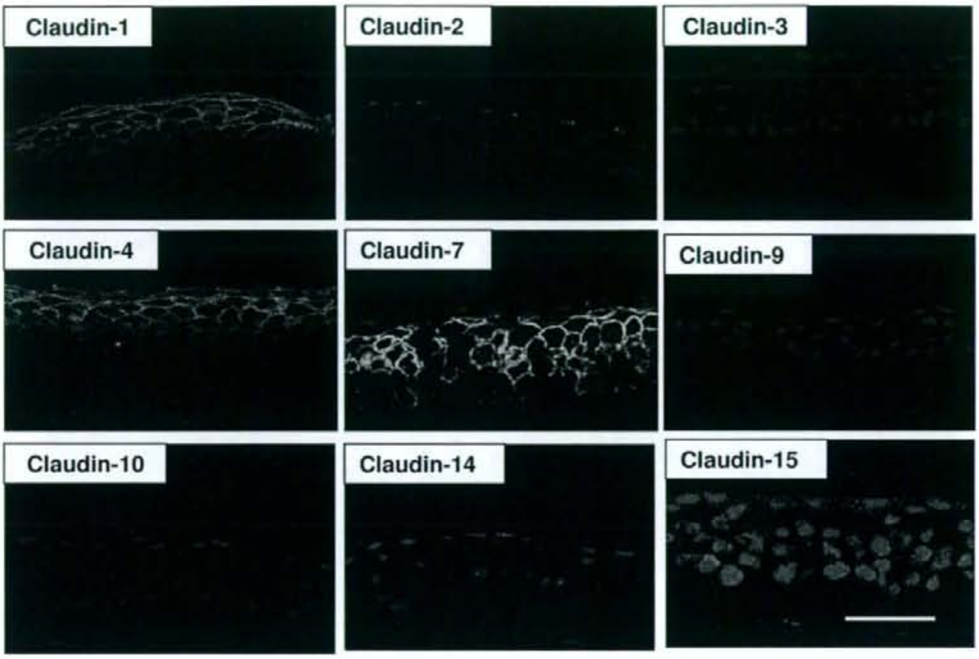
Claudin-1
Claudin-2
Claudin-3
Claudin-4
Claudin-7
Claudin-9
Claudin-10
Claudin-14
Claudin-15

Figure 3

Figure 4

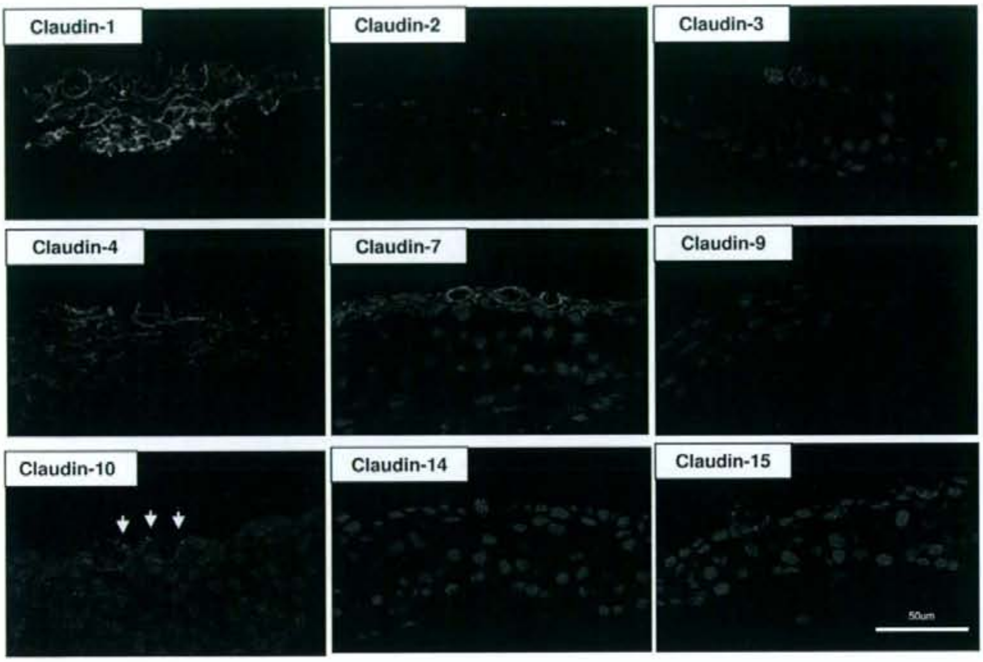
Claudin-1
Claudin-2
Claudin-3
Claudin-4
Claudin-7
Claudin-9
Claudin-10
Claudin-14
Claudin-15
50um

Figure 5

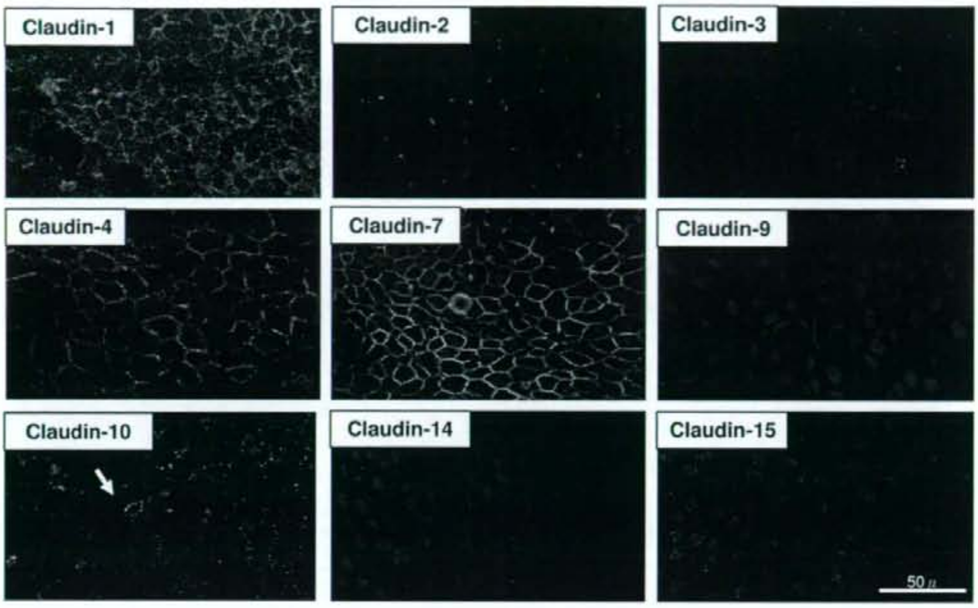
Claudin-1
Claudin-2
Claudin-3
Claudin-4
Claudin-7
Claudin-9
Claudin-10
Claudin-14
Claudin-15
50μ

Figure 6

Units of the Burnside Ring That Are Not Represented by Linear G-spheres

Ikumitsu NAGASAKI

STUDIA HUMANA et NATURALIA

No. 42

京都府立医科大学医学部医学科

Kyoto Prefectural University of Medicine

2008年12月　抜刷

UNITS OF THE BURNSIDE RING THAT ARE NOT REPRESENTED BY LINEAR G-SPHERES

IKUMITSU NAGASAKI

ABSTRACT. After recalling the Burnside ring $A(G)$ of a compact Lie group G, we show that for an arbitrary nonsolvable compact Lie group G, there exists a unit of $A(G)$ represented by a semilinear G-sphere, but not by a linear G-sphere. This is a refinement of Bauer's result, and also generalizes Matsuda's result.

1. THE BURNSIDE RING OF A COMPACT LIE GROUP

Throughout this paper, a subgroup means a closed subgroup unless otherwise stated. We denote by $H \leq K$ that H is a subgroup of K, and by $H < K$ that H is a proper subgroup of K. As usual (H) denotes the conjugacy class of H and $(H) \leq (K)$ means that H is subconjugate to K.

We shall begin by recalling the Burnside ring. General references are tom Dieck [5, 6], Fausk [7], etc. The Burnside ring $A(G)$ of a compact Lie group G is the ring consisting of equivalence classes of finite G-CW complexes (or compact smooth G-manifolds) with respect to the following equivalence relation $\sim$:

$$X \sim Y \text{ if and only if } \chi(X^H) = \chi(Y^H) \text{ for every subgroup } H \text{ of } G,$$

where $\chi(-)$ denotes the Euler characteristic. The addition is given by disjoint union and the multiplication is given by cartesian product. Note that $A(G)$ is a commutative ring.

Originally the Burnside ring $\Omega(G)$ of a finite group G is defined as the Grothendieck ring of the category of finite G-sets and G-maps. Thus there are two definitions of the Burnside ring; however, one can see

Proposition 1.1. *For a finite group G, the homomorphism $i : \Omega(G) \to A(G)$ induced by regarding finite G-sets as 0-dimensional finite G-CW complexes is an isomorphism.*

Dedicated to Professor Yoko Suga on the occasion of her retirement from Kyoto Prefectural University of Medicine.

2000 *Mathematics Subject Classification.* Primary 57S15; Secondary 57S25.

Key words and phrases. Burnside ring, unit group, semilinear action, representation theory, transformation groups.

Remark. The Burnside ring of a compact Lie group is also defined by using homogeneous spaces G/H with finite $WH = NH/H$, see Fausk [7].

Let $s(G)$ be the space of subgroups of G with the Hausdorff metric induced from a bi-invariant metric on G. Then $s(G)$ is a countable, compact metric space. Let $\phi(G)$ be the space of conjugacy classes of subgroups H such that each $WH := NH/H$ is finite. We denote by $C(G)$ the ring of integer-valued continuous (or equivalently locally constant) functions on $\phi(G)$. The ring homomorphism $\chi_H : A(G) \to \mathbb{Z}$ is induced by setting $\chi_H([X]) = \chi(X^H)$, where X is a finite G-CW complex, and then the ring homomorphism $\varphi : A(G) \to C(G)$ is defined by setting $\varphi([X])(H) = \chi_H([X])$, $(H) \in \phi(G)$. The following is well known, see for example [6].

Proposition 1.2. *The homomorphism φ is injective.*

Via this injective ring homomorphism, one can regard $A(G)$ as a subring of $C(G)$.

Finally we recall some definitions in group theory. The derived group $G^{(1)}$ of G is defined as the closure of the commutator subgroup of G. Inductively $G^{(n)}$ is defined by $(G^{(n-1)})^{(1)}$. A compact Lie group G is called *perfect* if $G = G^{(1)}$ and *solvable* if $G^{(n)} = 1$ for some n.

Proposition 1.3. *Suppose that G is a connected compact Lie group. Then the following statements are equivalent.*

(1) *G is solvable.*
(2) *G is abelian.*
(3) *G is a torus.*

In particular, a (general) compact Lie group G is solvable if and only if the identity component G_0 is a torus and G/G_0 is a finite solvable group.

2. Representing units of the Burnside ring

Let V be an orthogonal representation space of G and SV the unit sphere of V. The action on SV is called a linear action and SV is called a linear G-sphere. A semilinear action on a sphere is a natural generalization of a linear action.

Definition. A smooth closed G-manifold Σ is called a semilinear G-sphere if, for every subgroup H of G, the H-fixed point set Σ^H is a homotopy sphere or the empty set.

Linear G-spheres are clearly semilinear G-spheres, but there are many semilinear G-spheres with nonlinear actions, see for example [10], [11].

Lemma 2.1. *Let Σ be a semilinear G-sphere. In the Burnside ring $A(G)$, the element $u = 1 - [\Sigma]$ is a unit of $A(G)$.*

Proof. In fact, $\varphi(u)(H) = 1 - \chi(\Sigma^H) = \pm 1$; therefore $\varphi(u)$ is a unit of $C(G)$ and so u is a unit of $A(G)$. □

Definition. If a unit u of $A(G)$ is described as $1 - [\Sigma]$ for a semilinear [resp. linear] G-sphere Σ, then u is called the unit realized by a semilinear [resp. linear] G-sphere.

Matsuda [4] studied the question of whether all units of the Burnside ring of a finite group G are represented by linear G-spheres, and showed that not all units are represented by linear G-spheres for an arbitrary finite nonsolvable group G. On the other hand, Bauer [1] showed that if G_0 is nonabelian, then there exists a unit of $A(G)$ represented by a G-homotopy representation, but not by a linear G-sphere. Here a G-homotopy representation is a G-CW version of a semilinear G-sphere; more precisely, it is a finite-dimensional G-CW complex X such that for every subgroup H, the H-fixed point set X^H is homotopy equivalent to a sphere S^n, $n = \dim X^H$, or the empty set. We note [10] that a G-homotopy representation is not necessarily realized by a semilinear G-sphere.

Our result is the following.

Theorem 2.2. *Let G be an arbitrary nonsolvable compact Lie group. There exists a unit of $A(G)$ represented by a semilinear G-sphere, but not by a linear G-sphere.*

We call a smooth G-disk D a *quasilinear* G-disk if, for every subgroup H, the H-fixed point set D^H is diffeomorphic to a disk or the empty set. One can easily check that the product of quasilinear G-disks with diagonal action is also a quasilinear G-sphere after smoothing corners, and note also that the boundary of a quasilinear G-disk is a semilinear G-sphere.

Lemma 2.3. *If D is a quasilinear G-disk, then the element $[D]$ represents an idempotent of $A(G)$.*

Proof. Since

$$\varphi([D])(H) = \chi(D^H) = \begin{cases} 1 & \text{if } D^H \neq \emptyset \\ 0 & \text{if } D^H = \emptyset, \end{cases}$$

we see that $\varphi([D])$ is an idempotent of $C(G)$ and so $[D]$ is an idempotent of $A(G)$. □

To show the theorem, we recall Oliver's work [12] in which he showed that if G is nonsolvable, then there exists a quasilinear G-disk D with the following property: For every subgroup H of G,

$$D^H = \begin{cases} \text{disk} & \text{if } H \text{ is solvable} \\ \emptyset & \text{if } H \text{ is nonsolvable.} \end{cases}$$

We call such a quasilinear G-disk a special G-disk.

Proof of Theorem 2.2. Take a special G-disk D and set $\Sigma = \partial D$. Replacing D by a special G-disk $D \times D \times I$, where I is the unit interval with trivial action, we may assume that $\dim D^H$ is odd for every nonsolvable subgroup H. By Lemmas 2.1 and 2.3, the element $[D]$ represents a nontrivial idempotent of $A(G)$ and $1 - [\Sigma]$ represents a unit of $A(G)$. Then

$$\begin{aligned} \varphi(1-[\Sigma])(H) &= (-1)^{\dim \Sigma^H + 1} \\ &= \begin{cases} -1 & \text{if } H \text{ is solvable,} \\ 1 & \text{if } H \text{ is nonsolvable.} \end{cases} \end{aligned}$$

On the other hand

$$\varphi(1-2[D])(H) = \begin{cases} -1 & \text{if } H \text{ is solvable,} \\ 1 & \text{if } H \text{ is nonsolvable.} \end{cases}$$

Thus it is seen that $1 - [\Sigma] = 1 - 2[D]$. The following proposition shows that the unit $1 - [\Sigma]$ is not represented by a linear G-sphere. □

Proposition 2.4. *Let e be a nontrivial idempotent of $A(G)$ (i.e., $e \neq 0, 1$). Then the unit $1 - 2e$ is not represented by a linear G-sphere.*

Proof. Suppose that $1 - 2e$ is represented by a linear G-sphere SV. We may assume that $e(1) = 0$ since $1 - 2(1 - e)$ is represented by $S(V \oplus \mathbf{R})$. Since e is nontrivial, there is a subgroup H such that $e(H) = 1$. Take a minimal subgroup H such that $e(H) = 1$. Since $(1 - 2e)(H) = (-1)^{\dim V^H}$, it follows that $\dim V^H$ is odd and $\dim V^K$ is even for every $K < H$.

We first consider the case where the subgroup H is connected. In this case H is not a torus, because if H is a torus, it follows that $e(H) = e(1) = 0$ since H is solvable by Proposition 1.3; this is a contradiction. Let T be a maximal torus of H; the normalizer $N_H(T)$ of T in H is, then, a proper subgroup of H, and $\dim V^K$ is even for every subgroup of $N_H(T)$. By the following proposition proved in the next section, $\dim V^H$ is even; this is a contradiction.

Proposition 2.5. *Let G be a connected compact Lie group and T a maximal torus of G. Let V be a representation of G. If $\dim V^K$ is even for every subgroup K of the normalizer $N_G(T)$ of T, then $\dim V^H$ is even for every subgroup H of G. In particular $\dim V^G$ is even.*

We next consider the case where H is disconnected. In this case H/H_0 is not hyperelementary, where H_0 is the identity component of H, because if H/H_0 is hyperelementary, then $e(H) = e(H_0)$ since H/H_0 is solvable; however $e(H_0) = 0$ and $e(H) = 1$ since H_0 is a proper subgroup of H; this is a contradiction. We consider V^{H_0} as an H/H_0-representation; then $\dim(V^{H_0})^{K/H_0} = \dim V^K$ is even for every hyperelementary subgroup K/H_0 of H/H_0. It follows from [9, (5.4.2)] that $\dim V^H$ $(= \dim(V^{H_0})^{H/H_0})$ is even; this is a contradiction. □

3. Proof of Proposition 2.5

In this section, we prove Proposition 2.5 using representation theory. We refer to Bröcker and tom Dieck [3] for notation and terminology of representation theory. Consider the irreducible decomposition of V:

$$V = a_1V_1 \oplus \cdots \oplus a_nV_n \oplus b\mathbf{R},$$

where a_i, b are non-negative integers and $\mathbf{R}$ denotes the trivial irreducible representation. Note that $\dim V_i > 1$. As is well known, there are 3 types of irreducible representations; namely complex type, real type and quaternionic type. If W is an irreducible orthogonal representation of complex type or quaternionic type, then the complification $W \otimes \mathbf{C}$ is isomorphic to $U \oplus \bar{U}$ or $U \oplus U$, respectively for some irreducible complex representation U, where $\bar{U}$ denotes the complex conjugate of U, see [3, II (6.6)]. Since

$$\dim W^H = \dim_{\mathbf{C}}(W \otimes \mathbf{C})^H = \dim_{\mathbf{C}}(U \oplus U)^H = \dim_{\mathbf{C}}(U \oplus \bar{U})^H,$$

in this case, it follows that $\dim W^H$ is even for every subgroup H. Therefore we may assume that all irreducible components are of real type. If W is an irreducible representation of real type, then $W \otimes \mathbf{C}$ is an irreducible complex representation; therefore we may assume that V is a complex representation, and thus it suffices to show the following.

Proposition 3.1. *Let G be a connected compact Lie group and T a maximal torus of G. Let V be a complex representation of G whose irreducible components are of real* *type. If* $\dim V^K$ *is even for every subgroup* K *of the normalizer* $N_G(T)$ *of* T*, then* $\dim V^H$ *is even for every subgroup* H *of* G*. Here* dim *means complex dimension.*

Proof. If G is a torus, then there is nothing to do. Assume that G is not a torus. Let

$$V = a_1V_1 \oplus \cdots \oplus a_nV_n \oplus b\mathbf{C}$$

be the irreducible decomposition of V such that V_i are of real type. We show that a_i, b are even. Let $\lambda_i \in LT^*$ be the dominant weight of V_i. Take a maximal weight among λ_i, say λ_1. Since $\bar{V}_1 \cong V_1$, there exists an element wT of the Weyl group $W_G(T)$ such that $w\lambda_1 = -\lambda_1$, see [3, VI (4.1)]. Let U_1 be the irreducible representation of T representing λ_1. Let I be the isotropy subgroup of λ_1 under the $N_G(T)$-action on LT^*. Then U_1 is considered as a representation of I. Set $K = \operatorname{Ker} U_1 < I$ and $L = \langle w, I \rangle \leq N_G(T)$. Since one can easily check that $K \lhd L$ and $I \lhd L$, there is an extension:

$$1 \to I/K \to L/K \to L/I \to 1.$$

Since $I/K \cong S^1$, $L/I \cong \mathbf{Z}/2$, and the L/I-action on I/K is nontrivial, it follows that L/K is isomorphic to Pin(2) or O(2). Consider V^K as a representation of L/K; then V^K decomposes into

$$V^K = c_1M_1 \oplus c_2M_2 \oplus \cdots \oplus c_rM_r \oplus c\mathbf{C}_- \oplus d\mathbf{C},$$

where M_i are 2-dimensional irreducible representations of L/K, and $\mathbf{C}_-$ is the nontrivial 1-dimensional irreducible representation. Since $\dim(V^K)^{L/K} = d$ and $\dim(V^K)^{I/K} = c + d$, it follows from assumption that c and d are even. Since the multiplicity of the dominant weight λ_1 is one, we have $c_1 = a_1$.

In the case of $L/K \cong$ Pin(2), since the irreducible representation M_1 has trivial kernel, M_1 is of quaternionic type. On the other hand, since V is of real type, V^K has a real structure. Hence, by [3, VI (4.7)], the integer a_1 must be even.

We next consider the case of $L/K \cong$ O(2). Let C_i be the kernel of the irreducible representation M_i. Note that C_i is finite cyclic, $C_1 = 1$, and $C_i \neq C_j$ if $i \neq j$. We may assume that if $C_i < C_j$, then $i < j$. Let D_i be a dihedral subgroup containing C_i as a normal subgroup of index 2. Note that $D_1 \cong \mathbf{Z}/2$. With these notations, we have $\dim M_i^{D_j} = 1$ if $C_j \leq C_i$ and $\dim M_i^{D_j} = 0$ if $C_j \leq C_i$. We inductively show that all c_i are even. Since $\dim(V^K)^{D_r} = c_r + d$, it follows that c_r is even. Assume that $c_{i+1}, \ldots, c_r$ are even. Since $\dim(V^K)^{D_i} = c_i + \epsilon_{i+1}c_{i+1} + \cdots + \epsilon_r c_r + d$, where ϵ_j is defined by $\epsilon_j = 1$ if $C_i \leq C_j$ and $\epsilon_j = 0$ if $C_i \leq C_j$, it follows that c_i is even, and hence $c_1 = a_1$ is even.

Thus we inductively conclude that every a_i is even. Finally since $\dim V$ is even, it follows that b is also even. Thus the proof is completed. □

REFERENCES

[1] S. Bauer, *Dimension functions of homotopy representations for compact Lie groups*, Math. Ann. **280** (1988), 247–285.

[2] G. E. Bredon, Introduction to compact transformation groups, Academic Press, 1972.

[3] T. Bröcker and T. tom Dieck, Representations of compact Lie groups, Graduate Texts in Mathematics **98**, Springer 1985.

[4] T. tom Dieck, *The Burnside ring of a compact Lie group. I*, Math. Ann. **215** (1975), 235-250.

[5] T. tom Dieck, Transformation groups and representation theory, Lecture Notes in Math. 766, Springer-Verlag, 1979.

[6] T. tom Dieck, Transformation groups, Walte de Gruyter & Co, 1987.

[7] H. Fausk, *Survey on the Burnside ring of compact Lie groups*, J. Lie theory **18** (2008), 351–368.

[8] E. Laitinen, *Unstable homotopy theory of homotopy representations*, Lecture Notes in Math. **1217** (1985), 210–248.

[9] T. Matsuda, *On units groups of Burnside rings*, Japan. J Math. **8** (1982), 71–93.

[10] I. Nagasaki, *Linearity of dimension functions for semilinear G-spheres*, Proc. Amer. Math. Soc. **130** (2002), 1843-1850.

[11] I. Nagasaki, *The Grothendieck group of spheres with semilinear actions for a compact Lie group*, Topol. and its Appl. **145** (2004) , 241-260.

[12] R. Oliver, *Smooth compact Lie group actions on disks*, Math. Z. **149** (1976), 71–96.

DEPARTMENT OF MATHEMATICS, KYOTO PREFECTURAL UNIVERSITY OF MEDICINE, 13 NISHITAKATSUKASA-CHO, TAISHOGUN, KITA-KU, KYOTO 603-8334, JAPAN

E-mail address: nagasaki@koto.kpu-m.ac.jp

Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 155 (2008) 1066–1076
Topology and its Applications
www.elsevier.com/locate/topol

Isovariant maps from free C_n-manifolds to representation spheres ☆

Ikumitsu Nagasaki [a,*], Fumihiro Ushitaki [b]

[a] Department of Mathematics, Kyoto Prefectural University of Medicine, 13 Nishitakatsukasa-cho, Taishogun Kita-ku, Kyoto 603-8334, Japan
[b] Department of Mathematics, Faculty of Science, Kyoto Sangyo University, Kamigamo Motoyama, Kita-ku, Kyoto 603-8555, Japan

Received 14 May 2007; received in revised form 22 January 2008; accepted 22 January 2008

Dedicated to Professor Kazuhiko Fukui on his sixtieth birthday

Abstract

The isovariant version of Borsuk–Ulam type theorems has been studied by Wasserman and the first author. In this paper, first we consider the relation between the existence of C_n-isovariant maps from free C_n-manifolds to representation spheres and Borsuk–Ulam type inequalities for their dimensions. Our main result classifies the C_n-isovariant maps by C_n-isovariant homotopy types when a Borsuk–Ulam type inequality holds. For proving it, we use the multidegree of a C_n-equivariant map developed by the first author.

MSC: primary 57S17; secondary 55M20, 55M35

Keywords: Borsuk–Ulam theorem; Isovariant map; Transformation groups; Cyclic group action

1. Introduction

Throughout this paper, all maps are understood to be continuous. Borsuk–Ulam theorem says that if there is a continuous map between spheres $f : S^m \to S^n$ such that $f(-x) = -f(x)$ for all $x \in S^m$, then $m \leqq n$ holds [1]. From a viewpoint of the theory of transformation groups, this theorem is read as follows.

Proposition 1.1. *Let C_2 be a cyclic group of order 2. Assume that C_2 acts on both S^m and S^n antipodally. If there exists a continuous C_2-map $f : S^m \to S^n$, then $m \leqq n$ holds.*

In the theory of equivariant topology, Borsuk–Ulam type theorems are one of the most basic theorems, since they state whether a G-equivariant map exists or not between given G-spaces. Recently one of such Borsuk–Ulam type results played an important role in a partial solution of the 11/8-conjecture [6]. Moreover, it has been generalized in various directions and applied in several areas of mathematics, for example, combinatorics [9], nonlinear analysis [13,14], and so on.

☆ This work was partially supported by Grant-in-Aid for Scientific Research ((C) No. 17540075), Japan Society for the Promotion of Science.
* Corresponding author.
E-mail addresses: nagasaki@koto.kpu-m.ac.jp (I. Nagasaki), ushitaki@ksuvx0.kyoto-su.ac.jp (F. Ushitaki).

0166-8641/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2008.01.007

Let G be a group. Suppose X and Y are G-spaces. A G-equivariant map $\varphi: X \to Y$ is called a *G-isovariant map*, if it preserves the isotropy groups, that is, $G_x = G_{\varphi(x)}$ holds for all $x \in X$. We note that if $\varphi: X \to Y$ is G-equivariant, it always holds that $G_x \subset G_{\varphi(x)}$ for each $x \in X$. For G-homotopy equivalent G-isovariant maps $\varphi, \psi: X \to Y$, let $F: X \times [0,1] \to Y$ be a G-homotopy from φ to ψ. If F is a G-isovariant map, it is called a *G-isovariant homotopy from φ to ψ*, and it is said that φ and ψ are *G-isovariantly homotopic*. We denote by $[X, Y]_G^{\mathrm{isov}}$ the G-isovariant homotopy classes of all G-isovariant maps from X to Y. Various theories and results in the isovariant setting have been obtained by several authors. For example, isovariant surgery theory on a stratified set [2,4], isovariant homotopy theory [5], the isovariant s-cobordism theorem [2,8], and so on. The following isovariant Borsuk–Ulam theorem was proved by Wasserman [15].

Proposition 1.2 *(Isovariant Borsuk–Ulam theorem). Let G be a finite solvable group, and let V and W be G-representations. If there exists a G-isovariant map $f: V \to W$, then the following inequality holds:*

$$\dim V - \dim V^G \leqq \dim W - \dim W^G,$$

where W^G is the G-fixed point set.

Various results concerning the relations between Borsuk–Ulam type inequalities and the existence of isovariant maps have been also studied by the first author [10–12]. In this paper, we begin by considering similar problem in the following situation: G is a finite group, the source space is a mod $|G|$-homology sphere on which G acts freely, where a mod$|G|$-homology sphere means a closed manifold whose homology groups are isomorphic to those of a sphere with $\mathbb{Z}/|G|$ coefficients. The target space is a unitary representation sphere; namely, the unit sphere of a unitary representation of G. Our first result is:

Theorem A. *Let G be a finite group, and M an m-dimensional* mod$|G|$*-homology sphere on which G acts freely. Let W be a unitary representation of G, and SW its G-representation sphere. If there exists a G-isovariant map $f: M \to SW$, then for any subgroup $H (\neq \{e\})$ of G, the inequality*

$$\dim M + 1 \leqq \dim SW - \dim SW^H$$

holds, where if $SW^H = \emptyset$, we put $\dim SW^H = -1$.

For a given G-action on SW, the *singular set $SW^{>1}$* is a subspace of SW defined by $SW^{>1} = \bigcup_{\{1\} \neq H \leqslant G} SW^H$. We obtain the following corollary immediately.

Corollary B. *Let G be a finite group, and M an m-dimensional* mod$|G|$*-homology sphere on which G acts freely. Let W be a unitary representation of G, and SW its G-representation sphere. If there exists a G-isovariant map $f: M \to SW$, then the inequality*

$$\dim M + 1 \leqq \dim SW - \dim SW^{>1} \tag{1}$$

holds, where if $SW^{>1} = \emptyset$, we put $\dim SW^{>1} = -1$.

We next consider the converse of Corollary B when G is finite cyclic. Let C_n denote a cyclic group of order n. Our second result is concerning the existence of C_n-isovariant maps under the inequality (1).

Theorem C. *Let M be an m-dimensional arcwise connected orientable closed C^∞-manifold with an orientation preserving free C_n-action, and W a faithful unitary representation space of C_n. If the inequality*

$$\dim M + 1 \leqq \dim SW - \dim SW^{>1} \tag{1}$$

holds, there exists a C_n-isovariant map $f: M \to SW$.

Under the assumption of Theorem C, we will discuss the classification problem of C_n-isovariant maps $f: M \to SW$ by C_n-isovariant homotopy types; in particular, we investigate the structure of $[M, SW]_{C_n}^{\mathrm{isov}}$ when the C_n-action on SW

is not free. We remark that if the action is free, $[M, SW]^{\mathrm{isov}}_{C_n} = [M, SW]_{C_n}$ holds. We will prove that $[M, SW]^{\mathrm{isov}}_{C_n}$ consists of exactly one element under the inequality $\dim M + 1 < \dim SW - \dim SW^{>1}$ by using equivariant obstruction theory.

On the other hand, if $\dim M + 1 = \dim SW - \dim SW^{>1}$, various types of C_n-isovariant maps from M to SW exist. In fact, set

$$\mathcal{A} = \{H \in \mathrm{Iso}(SW) \mid \dim SW^H = \dim SW^{>1}\},$$

then Corollary E says that there is a one to one correspondence between $[M, SW]^{\mathrm{isov}}_{C_n}$ and $\bigoplus_{H \in \mathcal{A}} \mathbb{Z}$. To construct this correspondence in Section 5, we will define the multidegree $\mathrm{mDeg}\, f$ which takes the value in $\bigoplus_{H \in \mathcal{A}} \mathbb{Z}$ for every C_n-map $f : M \to SW_{\mathrm{free}}$, where $SW_{\mathrm{free}} = SW \setminus SW^{>1}$. Theorem D in Section 5 is a Hopf-type theorem for the multidegrees of the C_n-maps and the key result for the classification problem of G-isovariant maps. Incidentally, the concept of a multidegree was originally introduced by the first author for showing the existence of the S^1-isovariant maps from a rational homology sphere to a representation sphere [11].

This paper is organized as follows. In Section 2, we prove Theorem A and Corollary B. In Section 3, we give a quick review from the equivariant obstruction theory in our setting which is used for proving Theorems C and D. Section 4 presents a proof of Theorem C. Section 5 is devoted to the explanation of our main results. After defining the multidegree of a C_n-map from M to SW_{free}, we state Theorem D which is a Hopf-type theorem for the multidegrees and Corollary E which is the solution of our classification problem, while their proofs are given in Section 6. In the last section, we illustrate a couple of examples.

2. Borsuk–Ulam type inequalities

In this section, we prove Theorem A. Throughout this section, G is a finite group, W is a unitary representation space of G, and M is a $C^\infty G$-manifold which satisfies the conditions given in the statement of Theorem A. For a subgroup H of G, let $(W^H)^\perp$ denote the orthogonal complement of W^H in W with respect to the Hermitian product. We note that $(W^H)^\perp$ is also a representation space of H.

Lemma 2.1. *Let p be a prime factor of $|G|$. If there exists a G-isovariant map $f : M \to SW$, there exists a C_p-equivariant map $f_p : M \to S((W^{C_p})^\perp)$.*

Proof. Since $f : M \to SW$ is C_p-isovariant and the action of C_p on M is free, it follows that $f(M) \subset SW \setminus SW^{C_p}$. The action of C_p on $S((W^{C_p})^\perp)$ is free, and there exists a C_p-homotopy equivalence between $SW \setminus SW^{C_p}$ and $S((W^{C_p})^\perp)$ given by the composition

$$SW \setminus SW^{C_p} \simeq_{C_p} W \setminus W^{C_p} = \left((W^{C_p})^\perp \setminus \{0\}\right) \times W^{C_p} \simeq_{C_p} (W^{C_p})^\perp \setminus \{0\} \simeq_{C_p} S\left((W^{C_p})^\perp\right).$$

Hence, we can construct a C_p-equivariant map $f_p : M \to S((W^{C_p})^\perp)$. □

Proof of Theorem A. By the homological assumption on M, it holds that $H_*(M; \mathbb{Z}/p) \cong H_*(S^m; \mathbb{Z}/p)$ for any prime factor p of $|G|$. By applying the C_p-Borsuk–Ulam theorem [7] to the C_p-map $f_p : M \to SW^{C_p}$ constructed in Lemma 2.1, we see that

$$\dim M \leqq \dim S\left((W^{C_p})^\perp\right) = \dim SW - \dim SW^{C_p} - 1.$$

Hence for any subgroup $H \neq \{e\}$, it holds that

$$\dim M + 1 \leqq \dim SW - \dim SW^H. \qquad □$$

Since $\dim SW^{>1} = \max_{\{e\} \neq H \leqslant G} \dim SW^H$, Corollary B is an immediate consequence of Theorem A.

3. Equivariant obstruction theory

For the convenience of the readers, we give a quick review of the equivariant obstruction theory in our setting. This theory is the main tool for proving Theorems C and D. The notations and propositions in this section are based on tom Dieck's book [3].

Let G be a finite group. We first introduce suitable equivariant cohomology groups in our setting. Let (X, A) be a relative G-CW complex such that G acts freely on $X \setminus A$, namely the n-skeleton X_n is obtained from X_{n-1} by attaching free n-cells. The filtration $(X_n \mid n \in \mathbb{Z})$ leads to a cellular chain complex $C_*(X, A)$

$$\cdots \to H_{n+1}(X_{n+1}, X_n) \xrightarrow{d} H_n(X_n, X_{n-1}) \to \cdots,$$

where homology groups are singular homology groups with coefficients in $\mathbb{Z}$ and d is the boundary homomorphism in the homology exact sequence for the triple $\{(X_{n+1}, X_n, X_{n-1})\}$.

For any integer $n \in \mathbb{Z}$, the G-action on X_n induces a G-action on $H_n(X_n, X_{n-1})$. Then, $H_n(X_n, X_{n-1})$ is a left $\mathbb{Z}G$-module and $C_*(X, A)$ becomes a chain complex of such modules. For any left $\mathbb{Z}G$-module π, the cochain complex

$$C_G^*(X, A; \pi) := \mathrm{Hom}_{\mathbb{Z}G}\big(C_*(X, A); \pi\big)$$

yields equivariant cohomology groups denoted by $\mathfrak{H}_G^*(X, A; \pi)$. If a left $\mathbb{Z}G$-module π is interpreted as a local coefficient system $\{\pi\}$ on $X/G \setminus A/G$, then it holds that

$$\mathfrak{H}_G^*(X, A; \pi) \cong H^*\big(X/G, A/G; \{\pi\}\big).$$

Let Y be an arcwise connected, n-simple G-space, where $n \geqq 1$, that is, $\pi_1(Y, y)$ acts trivially on $\pi_n(Y, y)$ for every base point y. Then homotopy groups with different base points are canonically isomorphic each other, so we denote this homotopy group by $\pi_n(Y)$. Moreover the canonical map $\pi_n(Y) \to [S^n, Y]$ is bijective. Since the action on Y induces the G-action on $[S^n, Y]$, the G-action on $\pi_n(Y)$ is also induced. Thus $\pi_n(Y)$ becomes a $\mathbb{Z}G$-module, and we have

$$\mathfrak{H}_G^*\big(X, A; \pi_n(Y)\big) \cong H^*\big(X/G, A/G; \{\pi_n(Y)\}\big).$$

In addition, if G acts trivially on $\pi_n(Y)$, we obtain

$$\mathfrak{H}_G^*\big(X, A; \pi_n(Y)\big) \cong H^*\big(X/G, A/G; \pi_n(Y)\big).$$

The following proposition shall be used for proving Theorems C and D.

Proposition 3.1. *(See [3].) Let G be a finite group, (X, A) a relative G-CW complex such that G acts freely on $X \setminus A$. Let Y be an arcwise connected m-simple G-space for every $1 \leqq m \leqq \dim(X \setminus A)$. Then, the following hold.*

(1) *If $\mathfrak{H}_G^q(X, A; \pi_{q-1}(Y)) = 0$ for every $1 \leqq q \leqq \dim(X \setminus A)$, then any G-map $f : A \to Y$ extends to a G-map $X \to Y$.*
(2) *When a G-map $f : A \to Y$ extends to two G-maps $F, F' : X \to Y$, if $\mathfrak{H}_G^q(X, A; \pi_q(Y)) = 0$ for every $1 \leqq q \leqq \dim(X \setminus A)$, then F and F' are G-homotopic.*

4. Existence of isovariant maps

Let M be an m-dimensional arcwise connected orientable closed C^∞-manifold with an orientation preserving free C_n-action, and W a unitary representation space of C_n. Set

$$\mathcal{A} = \big\{H \in \mathrm{Iso}(SW) \mid \dim SW^H = \dim SW^{>1}\big\},$$

where $\mathrm{Iso}(SW)$ is the set of isotropy groups of SW. We begin this section by proving the following lemma.

Lemma 4.1. *For distinct subgroups $H, H' \in \mathcal{A}$, we have:*

(1) *$\langle H, H' \rangle \notin \mathcal{A}$, where $\langle H, H' \rangle$ is the subgroup generated by H and H'.*
(2) *$W^H \neq W^{H'}$.*

Proof. First, we observe that $H_1 \subset H_2$ yields $H_1 = H_2$ for $H_1, H_2 \in \mathcal{A}$. In fact, assume $H_1 \subsetneqq H_2$. Then, it holds that $SW^{H_2} \subsetneqq SW^{H_1}$; thereby $\dim SW^{H_2} < \dim SW^{H_1}$ because the fixed point sets are spheres. By the definition of $\mathcal{A}$, we conclude that $H_2 \notin \mathcal{A}$, which contradicts the definition of H_2. (1) is its immediate consequence. For proving (2), we note that $W^H \cap W^{H'} = W^{\langle H, H' \rangle}$. Since $\langle H, H' \rangle \notin \mathcal{A}$, we see that $\dim SW^{\langle H,H' \rangle} < \dim SW^{>1} = \dim SW^H$. Thus we have $W^H \neq W^{H'}$. □

We put $SW_{\text{free}} = SW \setminus SW^{>1}$. Proposition 3.1 says that if $\mathfrak{H}^*_{C_n}(M; \pi_{*-1}(SW_{\text{free}})) = 0$, there exists a C_n-map $f : M \to SW_{\text{free}}$. Put $k = \dim SW - \dim SW^{>1}$. Since the C_n-representation is assumed to be unitary, we have $k \geqq 2$.

Lemma 4.2.

(1) *The space SW_{free} is arcwise connected, $(k-2)$-connected and m-simple for every $1 \leqq m \leqq k-1$.*
(2) *The cyclic group C_n acts trivially on $\pi_{k-1}(SW_{\text{free}})$.*

Proof. (1) The arcwise connectivity of SW_{free} follows from $k \geqq 2$. The $(k-2)$-connectivity follows from a general position argument. In fact, by the homotopy exact sequence, we see that $\pi_i(SW_{\text{free}}) \cong \pi_{i+1}(SW, SW_{\text{free}})$ for $i \leqq \dim SW - 2$. We show that $\pi_{i+1}(SW, SW_{\text{free}}) = 0$ for $i \leqq k-2$. Let $\alpha : (D^{i+1}, S^i) \to (SW, SW_{\text{free}})$ be any based map for $i \leqq k-2$. Since $\dim D^{i+1} + \dim SW^{>1} < \dim SW$, there exists a based map $\alpha' : (D^{i+1}, S^i) \to (SW, SW_{\text{free}})$ based homotopic to α such that $\alpha'(D^{i+1}) \cap SW^{>1} = \emptyset$. This implies that $\alpha'(D^{i+1}) \subset SW_{\text{free}}$; thereby $\pi_{i+1}(SW, SW_{\text{free}}) = 0$. Thus, if $k > 2$, SW_{free} is 1-connected; thereby it is m-simple for every $1 \leqq m \leqq k-1$. If $k = 2$, SW_{free} is 1-simple by Lemma 4.3 below.

(2) Since W is a unitary C_n-representation, the C_n-action on $\pi_{k-1}(SW_{\text{free}})$ can extend to an S^1-action on $\pi_{k-1}(SW_{\text{free}})$. Hence, the map induced by $g \in C_n$ is homotopic to the map induced by the identity element $e \in C_n$. Thus, C_n acts trivially on $\pi_{k-1}(SW_{\text{free}})$. □

Lemma 4.3. *If $k = 2$, then the fundamental group $\pi_1(SW_{\text{free}})$ is Abelian.*

Proof. Set

$$SW_{\mathcal{A}\text{-free}} = SW \setminus \bigcup_{H \in \mathcal{A}} SW^H = \bigcap_{H \in \mathcal{A}} (SW \setminus SW^H).$$

Since $\dim SW - \dim SW^K \geqq 4$ for $\{e\} \neq K \in \mathrm{Iso}(SW) \setminus \mathcal{A}$, we see that the inclusion $SW_{\text{free}} \subset SW_{\mathcal{A}\text{-free}}$ induces an isomorphism $\pi_1(SW_{\text{free}}) \cong \pi_1(SW_{\mathcal{A}\text{-free}})$ by a general position argument. Put

$$W_{\mathcal{A}\text{-free}} = W \setminus \bigcup_{H \in \mathcal{A}} W^H,$$

then $SW_{\mathcal{A}\text{-free}}$ is a strong deformation retract of $W_{\mathcal{A}\text{-free}}$. It is sufficient to show that $\pi_1(W_{\mathcal{A}\text{-free}})$ is Abelian. Since by assumption $\dim(W^H)^\perp = 2$ for $H \in \mathcal{A}$, it is irreducible. Moreover, since $(W^H)^\perp \neq (W^{H'})^\perp$ if $H \neq H'$, W is decomposed as

$$W = \bigoplus_{H \in \mathcal{A}} (W^H)^\perp \oplus W'$$

for some W'. Thus one can see that $W_{\mathcal{A}\text{-free}}$ is homeomorphic to $\prod_{H \in \mathcal{A}}((W^H)^\perp \setminus \{0\}) \times W'$ whose fundamental group is isomorphic to $\bigoplus_{H \in \mathcal{A}} \mathbb{Z}$. □

Proof of Theorem C. By Lemma 4.2, we have

$$\mathfrak{H}^*_G(M, \pi_{*-1}(SW_{\text{free}})) \cong H^*(M/C_n; \pi_{*-1}(SW_{\text{free}})). \tag{2}$$

By the $(k-2)$-connectivity of SW_{free}, we have $H^q(M/C_n; \pi_{q-1}(SW_{\text{free}})) = 0$ for $q \leqq k-1$. By Proposition 3.1, this shows the existence of a C_n-map f from M to SW_{free} both of which have free C_n-action. Since $f(M) \subset SW_{\text{free}}$, f is also a C_n-isovariant map from M to SW. □

5. Isovariant homotopy classes

In this section, we discuss the classification problem of C_n-isovariant maps between M and SW under the assumption of Theorem C. Since the C_n-action on M is assumed to be free,

$$[M, SW]^{\mathrm{isov}}_{C_n} = [M, SW_{\mathrm{free}}]_{C_n}$$

holds. For our purpose, we use equivariant obstruction theory and define the multidegree for a C_n-equivariant map. The obstruction to C_n-homotopy types lies in

$$\mathfrak{H}^*_{C_n}(M; \pi_*(SW_{\mathrm{free}})) \cong H^*(M/C_n; \pi_*(SW_{\mathrm{free}})).$$

Recall that $k = \dim SW - \dim SW^{>1}$.

Proposition 5.1. *If* $\dim M < k-1$, *then all* C_n*-isovariant maps from* M *to* SW *are* C_n*-isovariantly homotopic each other.*

Proof. Since SW_{free} is $(k-2)$-connected by Lemma 4.2, we have $H^q(M/C_n; \pi_q(SW_{\mathrm{free}})) = 0$ for $q \leqq k-2$. Hence the obstruction vanishes; thereby all C_n-isovariant maps from M to SW are C_n-isovariantly homotopic. □

Next, we consider the problem when $\dim M = k-1$. The cohomology group $H^q(M/C_n; \pi_q(SW_{\mathrm{free}}))$ vanishes if $q \neq k-1$ by a similar argument in the proof of Proposition 5.1. Thus, the obstruction lies in

$$\mathfrak{H}^{k-1}_{C_n}(M; \pi_{k-1}(SW_{\mathrm{free}})) \cong H^{k-1}(M/C_n; \pi_{k-1}(SW_{\mathrm{free}})).$$

Recall

$$\mathcal{A} = \{H \in \mathrm{Iso}(SW) \mid \dim SW^H = \dim SW^{>1}\}$$

and

$$SW_{\mathcal{A}\text{-free}} = SW \setminus \bigcup_{H \in \mathcal{A}} SW^H = \bigcap_{H \in \mathcal{A}} (SW \setminus SW^H).$$

Lemma 5.2. *Let* W *be a faithful unitary* C_n*-representation.*

(1) *The homomorphism* $i_*: \pi_{k-1}(SW_{\mathrm{free}}) \to \pi_{k-1}(SW_{\mathcal{A}\text{-free}})$ *induced by the inclusion map* $i: SW_{\mathrm{free}} \to SW_{\mathcal{A}\text{-free}}$ *is an isomorphism.*
(2) *The Hurewicz homomorphisms* $h: \pi_{k-1}(SW_{\mathcal{A}\text{-free}}) \to H_{k-1}(SW_{\mathcal{A}\text{-free}})$ *and* $h': \pi_{k-1}(SW_{\mathrm{free}}) \to H_{k-1}(SW_{\mathrm{free}})$ *are isomorphic.*

Proof. (1) By definition, we see that $\dim(SW_{\mathcal{A}\text{-free}} \setminus SW_{\mathrm{free}}) \leqq \dim SW^{>1} - 2 = \dim SW - k - 2$. Hence we have

$$\dim SW_{\mathcal{A}\text{-free}} - \dim(S^{k-1} \times I) - \dim(SW_{\mathcal{A}\text{-free}} \setminus SW_{\mathrm{free}}) \geqq 2.$$

Thus, by a general position argument, i_* is an isomorphism.

(2) By Lemma 4.2 and the Hurewicz theorem, h' is an isomorphism. By a similar argument to Lemma 4.2, $SW_{\mathcal{A}\text{-free}}$ is also arcwise connected and $(k-2)$-connected. Hence it follows from the Hurewicz theorem that h is an isomorphism. □

Lemma 5.3. *Let* W *be a faithful unitary* C_n*-representation. Then, there is an isomorphism*

$$\Phi: H_{k-1}(SW_{\mathrm{free}}) \to \bigoplus_{H \in \mathcal{A}} H_{k-1}(S(W^H)^\perp) \cong \bigoplus_{H \in \mathcal{A}} \mathbb{Z}$$

given by the following composite of isomorphisms:

$$H_{k-1}(SW_{\mathrm{free}}) \xrightarrow{i_*} H_{k-1}(SW_{\mathcal{A}\text{-free}}) \xrightarrow{j_*} \bigoplus_{H \in \mathcal{A}} H_{k-1}(SW \setminus SW^H) \xleftarrow{\oplus i_{H*}} \bigoplus_{H \in \mathcal{A}} H_{k-1}(S(W^H)^\perp) \cong \bigoplus_{H \in \mathcal{A}} \mathbb{Z},$$

where i, j *and* i_H *are inclusions.*

Proof. By investigating the structure of $H_{k-1}(SW_{\mathcal{A}\text{-free}})$, we will construct the isomorphism Φ. Put $\mathcal{A} = \{H_1, H_2, \dots, H_r\}$. We note that $SW_{\mathcal{A}\text{-free}} = \bigcap_{i=1}^{r}(SW \setminus SW^{H_i})$. Since $SW^{H_i} \cap SW^{H_r} = SW^{\langle H_i, H_r\rangle}$, we have $(\bigcap_{i=1}^{r-1}(SW \setminus SW^{H_i})) \cup (SW \setminus SW^{H_r}) = SW \setminus \bigcup_{i=1}^{r-1} SW^{\langle H_i, H_r\rangle}$. Consider the Mayer–Vietoris exact sequence:

$$\cdots \to H_k\left(SW \setminus \bigcup_{i=1}^{r-1} SW^{\langle H_i, H_r\rangle}\right) \to H_{k-1}(SW_{\mathcal{A}\text{-free}}) \to H_{k-1}\left(\bigcap_{i=1}^{r-1}(SW \setminus SW^{H_i})\right) \oplus H_{k-1}(SW \setminus SW^{H_r})$$
$$\to H_{k-1}\left(SW \setminus \bigcup_{i=1}^{r-1} SW^{\langle H_i, H_r\rangle}\right) \to \cdots.$$

Since the representation is unitary, Lemma 4.1 yields $\dim SW^{\langle H_i, H_r\rangle} \leqq \dim SW^{>1} - 2$ for each i $(1 \leqq i \leqq r-1)$. Hence we have

$$\dim SW - \dim \bigcup_{i=1}^{r-1} SW^{\langle H_i, H_r\rangle} \geqq \dim SW - (\dim SW^{>1} - 2) = k + 2,$$

thereby we see that $H_k(SW \setminus \bigcup_{i=1}^{r-1} SW^{\langle H_i, H_r\rangle}) = 0$ and $H_{k-1}(SW \setminus \bigcup_{i=1}^{r-1} SW^{\langle H_i, H_r\rangle}) = 0$. Thus we have

$$H_{k-1}(SW_{\mathcal{A}\text{-free}}) \cong H_{k-1}\left(\bigcap_{i=1}^{r-1}(SW \setminus SW^{H_i})\right) \oplus H_{k-1}(SW \setminus SW^{H_r}),$$

and by induction we see that the correspondence

$$H_{k-1}(SW_{\mathcal{A}\text{-free}}) \xrightarrow[\cong]{j_*} \bigoplus_{i=1}^{r} H_{k-1}(SW \setminus SW^{H_i})$$

is an isomorphism. As in the proof of Lemma 2.1, $SW \setminus SW^{H_i}$ is homotopy equivalent to $S((W^{H_i})^{\perp})$, which is a $(k-1)$-dimensional sphere because H_i belongs to $\mathcal{A}$. Therefore we have

$$\bigoplus_{i=1}^{r} H_{k-1}(SW \setminus SW^{H_i}) \cong \bigoplus_{i=1}^{r} H_{k-1}(S((W^{H_i})^{\perp})) \cong \bigoplus_{i=1}^{r} H_{k-1}(S^{k-1}) \cong \bigoplus_{i=1}^{r} \mathbb{Z}. \qquad \square$$

We are now prepared to define the multidegree of C_n-map $f : M \to SW_{\text{free}}$.

Definition 5.4. Let M be an orientable $(k-1)$-dimensional arcwise connected closed C^{∞}-manifold with an orientation preserving free C_n-action. Let W be a faithful unitary C_n-representation, and $f : M \to SW_{\text{free}}$ a C_n-map. Then, we define the *multidegree* of f denoted by $\operatorname{mDeg} f$ as

$$\operatorname{mDeg} f = \Phi(f_*[M]) \in \bigoplus_{H \in \mathcal{A}} \mathbb{Z},$$

where $[M]$ is the fundamental class of M.

Theorem D *(Hopf-type theorem). Under the assumption in Definition* 5.4, *the following hold.*

(1) $\operatorname{mDeg} : [M, SW_{\text{free}}]_{C_n} \to \bigoplus_{H \in \mathcal{A}} \mathbb{Z}$ *is injective.*
(2) *For C_n-maps $f, g : M \to SW_{\text{free}}$, it holds that* $\operatorname{mDeg} f - \operatorname{mDeg} g \in \bigoplus_{H \in \mathcal{A}} n\mathbb{Z}$.
(3) *Fix a C_n-map $f_0 : M \to SW_{\text{free}}$. Then for any $d \in \bigoplus_{H \in \mathcal{A}} n\mathbb{Z}$ there exists a C_n-map $f : M \to SW_{\text{free}}$ such that* $\operatorname{mDeg} f - \operatorname{mDeg} f_0 = d$.

Since $[M, SW]_{C_n}^{\text{isov}} = [M, SW_{\text{free}}]_{C_n}$, we obtain the following corollary, which is the classification theorem of C_n-isovariant maps in our setting.

Corollary E. *Let $f_0 : M \to SW$ be a fixed C_n-isovariant map. Then,* $\mathrm{mD}_{f_0} : [M, SW]^{\mathrm{isov}}_{C_n} \to \bigoplus_{H\in\mathcal{A}} \mathbb{Z}$ *defined by*

$$\mathrm{mD}_{f_0}([f]) = (\mathrm{mDeg}\, f - \mathrm{mDeg}\, f_0)/n$$

is a bijection.

6. Proof of the main result

In this section, we prove Theorem D presented in the previous section.

Lemma 6.1. *Let $f_0, f : M \to SW_{\mathrm{free}}$ be C_n-maps. Let $\gamma_{C_n}(f_0, f) \in \mathfrak{H}^{k-1}_{C_n}(M; \pi_{k-1}(SW_{\mathrm{free}}))$ denote the obstruction to the existence of a C_n-homotopy between f and f_0. Then, the assignment $[f] \in [M, SW_{\mathrm{free}}]_{C_n} \mapsto \gamma_{C_n}(f_0, f)$ gives a bijection*:

$$\rho_{C_n}(f_0) : [M, SW_{\mathrm{free}}]_{C_n} \to \mathfrak{H}^{k-1}_{C_n}\big(M; \pi_{k-1}(SW_{\mathrm{free}})\big).$$

Proof. See [3, II, (3.17)]. □

The map $\rho_{C_n}(f_0)$ defined in Lemma 6.1 depends on the choice of f_0 which is called a *reference map*. Set

$$C^*_{C_n}(M) = \mathrm{Hom}_{\mathbb{Z}C_n}\big(C_*(M), \pi_{k-1}(SW_{\mathrm{free}})\big)$$

and

$$C^*(M) = \mathrm{Hom}_{\mathbb{Z}}\big(C_*(M), \pi_{k-1}(SW_{\mathrm{free}})\big).$$

We define a homomorphism $\bar{\tau} : C^{k-1}(M) \to C^{k-1}_{C_n}(M)$ by $\bar{\tau}(f)(c) = \sum_{g\in C_n} gf(g^{-1}c)$, where $f \in C^{k-1}(M)$ and $c \in C_{k-1}(M)$. Then, by [3, II, pp. 123–124], $\bar{\tau}$ induces the norm homomorphism

$$\tau : H^{k-1}\big(M; \pi_{k-1}(SW_{\mathrm{free}})\big) \to \mathfrak{H}^{k-1}_{C_n}\big(M; \pi_{k-1}(SW_{\mathrm{free}})\big).$$

By forgetting the C_n-action, the forgetful map

$$\varepsilon : \mathfrak{H}^{k-1}_{C_n}\big(M; \pi_{k-1}(SW_{\mathrm{free}})\big) \to H^{k-1}\big(M; \pi_{k-1}(SW_{\mathrm{free}})\big)$$

is defined.

Lemma 6.2. *The composition $\tau \circ \varepsilon : \mathfrak{H}^{k-1}_{C_n}(M; \pi_{k-1}(SW_{\mathrm{free}})) \to \mathfrak{H}^{k-1}_{C_n}(M; \pi_{k-1}(SW_{\mathrm{free}}))$ is multiplication by n.*

Proof. See [3, II, p. 124]. □

Lemma 6.3. *The forgetful map $\varepsilon : \mathfrak{H}^{k-1}_{C_n}(M; \pi_{k-1}(SW_{\mathrm{free}})) \to H^{k-1}(M; \pi_{k-1}(SW_{\mathrm{free}}))$ is injective.*

Proof. By Eq. (2), $\mathfrak{H}^*_G(M, \pi_{*-1}(SW_{\mathrm{free}})) \cong H^*(M/C_n; \pi_{*-1}(SW_{\mathrm{free}}))$. Since $\dim M/C_n = k-1$ and the action of C_n on M is orientation preserving, we obtain that

$$\mathfrak{H}^{k-1}_{C_n}\big(M; \pi_{k-1}(SW_{\mathrm{free}})\big) \cong \pi_{k-1}(SW_{\mathrm{free}}) \cong \bigoplus_{H\in\mathcal{A}} \mathbb{Z}.$$

Since this cohomology group is torsion free, it follow from Lemma 6.2 that $\tau \circ \varepsilon$ is injective; thereby, ε is injective. □

Lemma 6.4. *The norm homomorphism*

$$\tau : H^{k-1}\big(M; \pi_{k-1}(SW_{\mathrm{free}})\big) \to \mathfrak{H}^{k-1}_{C_n}\big(M; \pi_{k-1}(SW_{\mathrm{free}})\big)$$

is an isomorphism.