4.1.1 Firm x chooses $x = 0$

When firm x locates at the center, firm y can either chooses to locate off the center and sets its price optimally given p_x or it can choose to locate at the center. If firm y locates at the centre, its optimal strategy is to undercut p_x marginally whenever $p_x \leq \frac{V}{2}$. For $p_x > \frac{V}{2}$, the optimal $p_y = \frac{V}{2}$.

local optimum with $y > x = 0$ Note that with $x = 0$, for consumers from the other two rays to purchase from firm y, it must be the case that even the consumer located at the center purchases from firm y. Since x is the consumer located at 0's ideal bundle, if this consumer buys from y, it implies that $D_x = 0$. This clearly would not occur in equilibirum. First, even if $p_y = 0$, firm x can still charge a low but positive price and induce the consumer located at the center to buy its good. Second, with D_y coming from the other two rays and $D_x = 0$, firm y's best location would be to locate at $y = 0$. Therefore, for the analysis here, we only focus on the case that D_y comes only from r_y.

To characterise D_y, we define two critical p_y levels. Let p_1 be the price such that $\bar{t}_y = 1$.

$$V - \tau\left(\bar{t}_y - y\right) - p_1 = 0$$

$$\bar{t}_y = \frac{V - p_1}{\tau} + y = 1 \Leftrightarrow p_1 \equiv V - \tau(1 - y) \tag{1}$$

The second critical value is the p_y such that $\underline{t}_y = \bar{t}_x$. Let this price level be p_2.

$$y - \frac{V - p_2}{\tau} = \frac{V - p_x}{\tau} \Leftrightarrow p_2 \equiv 2V - y\tau - p_x. \tag{2}$$

Note that $p_1 \geq p_2$ if $p_x \geq V - 2y\tau + \tau$. For $p_x < V - 2y\tau + \tau$, $p_1 \leq p_2$:

$$\begin{aligned} D_y &= 1 - t_{xy} & p_y &\leq p_1 \\ D_y &= \bar{t}_y - t_{xy} & p_1 \leq p_y &\leq p_2 \\ D_y &= \bar{t}_y - \underline{t}_y & p_y &\geq p_2. \end{aligned}$$

For $p_x \geq V - 2y\tau + \tau$, $p_1 \geq p_2$:

$$\begin{aligned} D_y &= 1 - t_{xy} & p_y &\leq p_2 \\ D_y &= 1 - \underline{t}_y & p_2 \leq p_y &\leq p_1 \\ D_y &= \bar{t}_y - \underline{t}_y & p_y &\geq p_1. \end{aligned}$$

Given $(p_x, x = 0)$, firm y charges the local monopoly price, $\frac{V}{2}$, if there is enough space left on r_y. Or if

$$1 - \bar{t}_x = 1 - \frac{V - p_x}{\tau} \geq \frac{V}{\tau} \Leftrightarrow p_x \geq 2V - \tau.$$

For $p_x < 2V - \tau$, the two firms are in competition and in equilibirum, all consumers on r_y purchase one unit of good.

Remark 1 *For $y > x = 0$ with D_y coming only from r_y, firm y's local optimal location is to locate at $\bar{t}_x + \frac{V}{2\tau} \leq y \leq 1 - \frac{V}{2\tau}$ if $p_x \geq 2V - \tau$ and $\bar{t}_y = 1$ otherwise.*

Consider the case when $\bar{t}_y \geq 1$ and $\underline{t}_y < \bar{t}_x$, t_{xy} is defined by

$$V - \tau(y - t_{xy}) - p_y = V - \tau t_{xy} - p_x \Leftrightarrow t_{xy} = \frac{\tau y + p_y - p_x}{2\tau}.$$

The demand for firm y is

$$D^y = 1 - t_{xy} = 1 - \frac{\tau y + p_y - p_x}{2\tau}.$$

The demand which comes from the further end of the market is $1 - y$ and the demand which comes from the end closer to the centre is

$$y - t_{xy} = \frac{\tau y - p_y + p_x}{2\tau}.$$

For any given p_y, by moving closer to the centre, the marginal gain of market from the far end is 1 while the marginal loss of market due to more competition with firm x is $-\frac{1}{2}$. Therefore firm y always has the incentive to move closer to firm x. On the other hand, when $\bar{t}_y = 1$ and $\underline{t}_y \leq \bar{t}_x$, by moving closer to the centre, the marginal loss of demand is -1 while the marginal gain of demand is $\frac{1}{2}$. Therefore, the optimal location is the one such that $\bar{t}_y = 1$.

For $p_x < 2V - \tau$, y always locates in the position such that $\bar{t}_y = 1$.

$$\bar{t}_y = 1 \Leftrightarrow V - \tau(1 - y) - p_y = 0.$$

This gives

$$y = \frac{\tau - V + p_y}{\tau},$$

and

$$\begin{aligned} D^y &= 1 - t_{xy} = 1 - \frac{\tau y + p_y - p_x}{2\tau} = 1 - \frac{\tau \frac{\tau - V + p_y}{\tau} + p_y - p_x}{2\tau} \\ &= \frac{V + \tau + p_x - 2p_y}{2\tau}. \end{aligned}$$

Firm y solves

$$\max_{p_y} \pi^y = \frac{V + \tau + p_x - 2p_y}{2\tau} p_y.$$

The FOC gives the optimal price and location,

$$p_y = \frac{V + \tau + p_x}{4} \text{ and } y = \frac{5\tau - 3V + p_x}{4\tau}.$$

Note that $p_1 \geq p_2$ if $p_x \geq V - 2y\tau + \tau$. Substituting in the equilibrium price and location, $p_1 \geq p_2$ if

$$p_x \geq V - 2y\tau + \tau = V - 2\frac{5\tau - 3V + p_x}{4\tau}\tau + \tau.$$

Or if

$$p_x \geq \frac{5}{3}V - \tau.$$

Since $\frac{5}{3}V - \tau < 2V - \tau$, for $\frac{5}{3}V - \tau \leq p_x < 2V - \tau$, $p_1 > p_2$.

For $p_x < \frac{5}{3}V - \tau$ and $p_1 < p_2$, $D_y = 1 - t_{xy}$. For $p_x \geq \frac{5}{3}V - \tau$, $p_1 \geq p_2$. Therefore, for $\frac{5}{3}V - \tau \leq p_x \leq 2V - \tau$, the solution is a corner solution with the optimal price and location determined by $\bar{t}_y = 1$ and $\underline{t}_y = \bar{t}_x$, or $p_y = p_1 = p_2$. This gives

$$y = \frac{V - p_x + \tau}{2\tau},\ p_y = \frac{3V - \tau - p_x}{2},$$

and

$$\pi_y = \frac{(\tau - V + p_x)(3V - \tau - p_x)}{2\tau}.$$

We summarise the firm y's best response in the following lemma.

Lemma 1 *If $x = 0$, $y > 0$ and D_y comes only from r_y, firm y's local best response is:*

$$p_y = \begin{cases} \frac{V+\tau+p_x}{4} \\ \frac{3V-\tau-p_x}{2} \\ \frac{V}{2} \end{cases},\quad y = \begin{cases} \frac{5\tau-3V+p_x}{4\tau} \\ \frac{\tau+V-p_x}{2\tau} \\ y \geq \bar{t}_x + \frac{V}{2\tau} \end{cases},$$

$$\pi^y = \begin{cases} \frac{(V+\tau+p_x)^2}{16\tau} & p_x \leq \frac{5}{3}V - \tau \\ \frac{(\tau-V+p_x)(3V-\tau-p_x)}{2\tau} & \frac{5}{3}V - \tau < p_x < 2V - \tau \\ \frac{V^2}{2\tau} & p_x \geq 2V - \tau \end{cases}.$$

Note that we need the additional restriction that $y \geq 0$.

$$\frac{5\tau - 3V + p_x}{4\tau} \geq 0 \Longrightarrow p_x \geq 3V - 5\tau.$$

$$\frac{5}{3}V - \tau \geq 3V - 5\tau \text{ if } V \leq 3\tau.$$

For $p_x < 3V - 5\tau$, firm y's optimal response would be $y = 0$.

local optimum with $y = 0$ Given $x = 0$, if firm y chooses $y > 0$ and p_y so that it sells to other rays, as discussed previously, this implies $D_x = 0$. The optimal location is therefore $y = 0$. For any given p_y, by moving from $y > 0$ closer to $y = 0$, the increase in D_y comes from the other two rays while the decrease in D_y is only from r_y. Note that if we compare two equilibrium: $(x = 0, y = 0)$ and $(x = 0, y > 0)$, it may be the case that $\pi_y (y = x = 0) > \pi_x (x = 0, y > 0)$. By undercutting firm x a little bit, firm y practically gets the same profit as firm x, except that there is no additional firm out there. Therefore, as long as $v (t_{xy}) > 0$, $\pi_y (y = x = 0) > \pi_x (x = 0, y > 0)$.

Lemma 2 *When $x = 0$, y's local best response so that D_y comes from all rays is $y = 0$ and*

$$p_y = \begin{cases} \textit{Not defined} & p_x \leq \frac{V}{2} \\ \frac{1}{2}V & p_x > \frac{V}{2} \end{cases};$$

$$\pi^y = \begin{cases} \textit{Not defined} & p_x \leq \frac{V}{2} \\ \frac{3V^2}{4\tau} & p_x > \frac{V}{2} \end{cases}.$$

For $p_x \leq \frac{V}{2}$, firm y would like to charge p_y as close to p_x but not equal to p_x. For continuous price setting, the best response is not defined.

4.1.2 Firm x chooses $x > 0$

We analyse in this section the case where firm x chooses $x > 0$. We first argue that the following two cases would never occur in equilibrium: (1) $y > x$ and $r_y = r_x$; (2) $y > 0$ and $r_y \neq r_x$. Therefore, the only case we need to consider is $x \geq y \geq 0$ and $r_y = r_x$. These three cases completes the analysis for $x > 0$.

1. $y > x$ and $r_y = r_x$ is never optimal

   For $x > 0$ and $r_y = r_x$, firm y would never locate at $y > x > 0$. Apart from the trivial case where both firms act as monopolists, for any given p_y, firm y is better off by locating at $r_y \neq r_x$.

2. $x > 0$, $y > 0$ and $r_y \neq r_x$ is never an equilibrium

   If $y > 0$, $r_y \neq r_x$, and t_{xy} locates on r_x, then by moving towards 0, the loss of demand from r_y is compensated by the gain of demand from r_i, $r_i \neq r_x \neq r_y$. Therefore, total demand from $r_y \neq r_x$ remains the same while demand from r_x increases. Thus, it is never optimal to choose $y > 0$ and $r_y \neq r_x$. If $y > 0$, $r_y \neq r_x$, and t_{xy} locates on r_y, then by the same argument, it is never optimal for x to choose $x > 0$.

Therefore, the only case we need to analyse here is $x \geq y \geq 0$ and $r_y = r_x$. We proceed by analysing D_y and optimal (p_y, y). To facilitate our discussion of D_y, we first define a few critical price levels:

$$\underline{p}_y \text{ solves } \bar{t}_y = \underline{t}_x \Rightarrow \underline{p}_y = 2V - \tau(x-y) - p_x.$$
$$\tilde{p}_y \text{ solves } \underline{t}_y = 1 \Rightarrow \tilde{p}_y = V - \tau(1+y).$$
$$p_y^0 \text{ solves } \underline{t}_y = 0 \Rightarrow p_y^0 = V - \tau y.$$

By definition, $\tilde{p}_y < p_y^0$ and $\tilde{p}_y < \underline{p}_y$. $\underline{p}_y \geq p_y^0$ if $y \geq \frac{x\tau - V + p_x}{2\tau}$.

Note first that $p_y \geq p_y^0$ would never occur in equilibrium. If this is the case, given any price, y is better off locating at $r_y \neq r_x$. We analyse the cases for $p_y \leq p_y^0$ here. Therefore, for $y \leq \frac{x\tau - V + p_x}{2\tau}$, $p_y^0 \geq \underline{p}_y > \tilde{p}_y$, and the demand for firm y is

$$D_y = \begin{cases} 2 + t_{xy} = \frac{4\tau + \tau(x+y) + p_x - p_y}{2\tau} & p_y \leq \tilde{p}_y \\ 2\underline{t}_y + t_{xy} = \frac{4V + p_x - 5p_y + \tau(x-3y)}{2\tau} & \tilde{p}_y \leq p_y \leq \underline{p}_y \\ \bar{t}_y + 2\underline{t}_y = 3\frac{V-p_y}{\tau} - y & \underline{p}_y \leq p_y \leq p_y^0 \end{cases}.$$

For $y \geq \frac{x\tau - V + p_x}{2\tau}$ we have $\underline{p}_y \geq p_y^0 > \tilde{p}_y$, and

$$D_y = \begin{cases} 2 + t_{xy} = \frac{4\tau + \tau(x+y) + p_x - p_y}{2\tau} & p_y \leq \tilde{p}_y \\ 2\underline{t}_y + t_{xy} = \frac{4V + p_x - 5p_y + \tau(x-3y)}{2\tau} & \tilde{p}_y \leq p_y \leq p_y^0 \end{cases}$$

First, notice that for the demand specification, firm y would only prefer to choose a bigger y in the region where the (y, p_y) combination gives $p_y(y) \leq \tilde{p}_y$. In other cases, firm y would like to locate as close to the centre as possible. This gives $y^* = 0$.

We discuss each of the demand specifications in turn.

Case 1 For $p_y \leq \tilde{p}_y$, π_y increases in y. Firm y would like to increase y provided that the conditions $y \leq x$ and $p_y \leq \tilde{p}_y$, or equivalently, $y \leq \frac{V-\tau-p_y}{\tau}$, are satisfied. Therefore, $y^* = \min\left\{x, \frac{V-\tau-p_y}{\tau}\right\}$.

$$x \leq \frac{V-\tau-p_y}{\tau} \text{ if } p_y \leq V - \tau(1+x).$$

Therefore, the optimal (y, p_y) combination is that for $p_y \leq V - \tau(1+x)$, firm y locates at $y = x$. For $p_y > V - \tau(1+x)$, the optimal (p_y, y) satisfies $p_y = \tilde{p}_y$ or $y = \frac{V - p_y - \tau}{\tau}$.

For $p_y \leq V - \tau(1+x)$ and $y = x$,

$$D_y = \begin{cases} 3 & \text{if } p_y < p_x \\ 0 & \text{if } p_y > p_x. \end{cases}$$

We assume that firms share the market if they charge the same price. This case, however, will not occur in equilibrium. For $p_y \leq V - \tau(1+x)$, the local best response is $p_y = \min\{V - \tau(1+x), \text{ Not defined}\}$. For the not defined part of the best response, firm y would like to charge p_y as close to p_x as possible with $p_y < p_x$. As noted previously, for the model with continuous price setting, the best response is not defined.

For $p_y > V - \tau(1+x)$, $p_y = \tilde{p}_y = V - \tau(1+y)$. This gives

$$\pi_y = \left(\frac{3\tau + \tau x + V + p_x - 2p_y}{2\tau}\right) p_y.$$

The FOC gives

$$p_y = \frac{3\tau + \tau x + V + p_x}{4}.$$

Note that with the restriction $y \geq 0$, the solution is only relevant for

$$p_y \leq V - \tau.$$

Or

$$p_x \leq 3V - 7\tau - x\tau.$$

For $p_y \geq V - \tau$, the corner solution occurs at $y = 0$ and $p_y = V - \tau$.

The solution is interior if

$$\frac{3\tau + \tau x + V + p_x}{4} > V - \tau(1+x)$$

or if

$$p_x > 3V - 7\tau - 5\tau x.$$

Compare the critical values:

$$3V - 7\tau - 5\tau x \geq V - \tau(1+x)$$

if

$$x \leq \frac{V - 3\tau}{2\tau}.$$

For $x \leq \frac{V-3\tau}{2\tau}$, the local best response in the area $p_y \leq \widetilde{p}_y$, is

$$p_y = \begin{cases} \text{Not defined} & p_x \leq V - \tau(1+x); \\ V - \tau(1+x) & V - \tau(1+x) < p_x \leq 3V - 7\tau - 5\tau x; \\ \frac{3\tau + \tau x + V + p_x}{4} & 3V - 7\tau - 5\tau x \leq p_x \leq 3V - 7\tau - x\tau; \\ V - \tau & p_x \geq 3V - 7\tau - x\tau. \end{cases}$$

For $x \geq \frac{V-3\tau}{2\tau}$, the local best responses are

$$p_y = \begin{cases} \text{Not defined} & p_x \leq 3V - 7\tau - 5\tau x; \\ \frac{3\tau + \tau x + V + p_x}{4} & 3V - 7\tau - 5\tau x \leq p_x \leq 3V - 7\tau - x\tau; \\ V - \tau & p_x \geq 3V - 7\tau - x\tau. \end{cases}$$

Case 2 For $\tilde{p}_y \leq p_y \leq \underline{p}_y$ and $\tilde{p}_y \leq p_y \leq p_y^0$,

$$\pi_y = \left(\frac{4V + p_x - 5p_y + \tau(x - 3y)}{2\tau}\right) p_y.$$

Firm y's profit, π_y, decreases in y in this region and firm y would like to locate as close to the centre as possible. Given that the boundaries of this region is defined by $\tilde{p}_y \leq p_y \leq \underline{p}_y$ and $\tilde{p}_y \leq p_y \leq p_y^0$, the optimal location is y^* satisfies $p_y = \underline{p}_y$ if $p_y > 2V - \tau x - p_x$, $y^* = 0$ if $V - \tau \leq p_y \leq 2V - \tau x - p_x$ and y^* satisfies $p_y = \widetilde{p}_y$ if $p_y \leq V - \tau$. Note that the part where $p_y = \underline{p}_y$ is only relevant for $y \leq \frac{x\tau - V + p_x}{2\tau}$.

For $p_y \leq V - \tau$, the local optimum satisfies $p_y = \widetilde{p}_y$ with the local best response $p_y = \frac{3\tau + \tau x + V + p_x}{4}$ and $y^* = \frac{-7\tau + 3V - x\tau - p_x}{4\tau}$. With $y \geq 0$, the local best response is relevant for $p_x \leq 3V - 7\tau - x\tau$. For $p_x \leq 3V - 7\tau - x\tau$, $\frac{3\tau + \tau x + V + p_x}{4} \leq V - \tau$.

For $V - \tau \leq p_y \leq 2V - \tau x - p_x$, the local optimal location is $y = 0$ with

$$\pi_y = \left(\frac{4V + p_x - 5p_y + \tau x}{2\tau}\right) p_y.$$

The FOC gives the local best response

$$p_y = \frac{(4V + x\tau + p_x)}{10}.$$

$$\frac{(4V + x\tau + p_x)}{10} \geq V - \tau \text{ if } p_x \geq 6V - 10\tau - x\tau.$$

$$\frac{(4V + x\tau + p_x)}{10} \leq 2V - \tau x - p_x$$

if

$$p_x \leq \frac{16V - 11x\tau}{11}.$$

$$\frac{16V - 11x\tau}{11} \geq 6V - 10\tau - x\tau \text{ if } V \leq \frac{11}{5}\tau \approx 2.2\tau.$$

For $p_y > 2V - \tau x - p_x$, the optimal y satisfies $p_y = \underline{p}_y$ or

$$y = \frac{x\tau - 2V + p_x + p_y}{\tau}.$$

This gives the profit

$$\pi_y = \frac{(5V - x\tau - p_x - 4p_y)\, p_y}{\tau}.$$

The FOC gives

$$p_y = \frac{5V - x\tau - p_x}{8}.$$

$$\frac{5V - x\tau - p_x}{8} \geq 2V - \tau x - p_x \text{ if } p_x \geq \frac{11V - 7x\tau}{7}.$$

Check the boundary values:

$$6V - 10\tau - x\tau \geq 3V - 7\tau - x\tau \text{ if } V \geq \tau.$$

$$\frac{16V - 11x\tau}{11} \geq 3V - 7\tau - x\tau \text{ if } V \leq \frac{77}{17}\tau \approx 4.53\tau.$$

$$\frac{11V - 7x\tau}{7} \geq 3V - 7\tau - x\tau \text{ if } V \leq \frac{49}{10}\tau.$$

$$\frac{11V - 7x\tau}{7} \geq 6V - 10\tau - x\tau \text{ if } V \leq \frac{70}{31}\tau \approx 2.2581\tau.$$

Note that

$$\frac{11V - 7x\tau}{7} \geq \frac{16V - 11x\tau}{11}.$$

For $V \leq \tau$, since the region $p_y \leq V - \tau$ is not relevant, the local best response is

$$p_y = \begin{cases} \frac{(4V+x\tau+p_x)}{10} & \text{if } 6V - 10\tau - x\tau \leq p_x \leq \frac{16V-11x\tau}{11} \\ 2V - \tau x - p_x & \text{if } \frac{16V-11x\tau}{11} \leq p_x \leq \frac{11V-7x\tau}{7} \\ \frac{5V-x\tau-p_x}{8} & \text{if } p_x \geq \frac{11V-7x\tau}{7} \end{cases}.$$

For $\tau \leq V \leq \frac{11}{5}\tau$, the local best response is

$$p_y = \begin{cases} \frac{3\tau+\tau x+V+p_x}{4} & \text{if } p_x \leq 3V - 7\tau - x\tau \\ V-\tau & \text{if } 3V-7\tau-x\tau \leq p_x \leq 6V-10\tau-x\tau \\ \frac{(4V+x\tau+p_x)}{10} & \text{if } 6V-10\tau-x\tau \leq p_x \leq \frac{16V-11x\tau}{11} \\ 2V-\tau x-p_x & \text{if } \frac{16V-11x\tau}{11} \leq p_x \leq \frac{11V-7x\tau}{7} \\ \frac{5V-x\tau-p_x}{8} & \text{if } p_x \geq \frac{11V-7x\tau}{7} \end{cases}.$$

For $\frac{11}{5}\tau \leq V \leq \frac{77}{17}\tau$, the local best response is

$$p_y = \begin{cases} \frac{3\tau+\tau x+V+p_x}{4} & \text{if } p_x \leq 3V-7\tau-x\tau \\ V-\tau & \text{if } 3V-7\tau-x\tau \leq p_x \leq V+\tau(1-x) \end{cases}.$$

For $V \geq \frac{77}{17}\tau$, the local best response is

$$p_y = \frac{3\tau+\tau x+V+p_x}{4}, y = \frac{-7\tau+3V-x\tau-p_x}{4\tau},$$

and

$$\pi_y = \frac{(V+3\tau+x\tau+p_x)^2}{16\tau}$$

for $p_x \leq V+\tau(1-x)$.

Case 3 Finally, for the case $\underline{p}_y \leq p_y \leq p_y^0$ the local best $y^* = 0$ with $p_y^* = \frac{V}{2}$. The solution is interior if $\frac{V}{2} \geq \underline{p}_y$. Or if

$$\frac{V}{2} \geq 2V - \tau(x-y) - p_x.$$

This holds for $p_x \geq \frac{3}{2}V - \tau x$. Note that this demand specification is only relevant for $p_x \geq V - \tau(x-2y)$.

Therefore, the local best response for $p_y \geq \underline{p}_y$ is

$$p_y = \begin{cases} 2V-\tau x-p_x & V-\tau x \leq p_x \leq \frac{3}{2}V-\tau x \\ \frac{V}{2} & p_x \geq \frac{3}{2}V-\tau x. \end{cases}$$

The optimal location is $y = 0$.

We are now ready to characterise the global best response for firm y. We compare the three different local best responses to get the global response. First, for $p_x \geq \frac{3}{2}V - \tau x$, the best response is $p_y = \frac{V}{2}$. This gives the unconstrained monopoly profit. Note that $\frac{16V-11x\tau}{11} \leq \frac{3}{2}V - \tau x < \frac{11V-7x\tau}{7}$.

Firm y's global best response 1. $V \leq \tau$

Note that $\widetilde{p}_y = V - \tau(1+y)$, for any $y \geq 0$, $\widetilde{p}_y \leq V - \tau \leq 0$. Therefore, the local best response for $p_y \leq \widetilde{p}_y$ is not relevant. We put together the local best responses for $\widetilde{p}_y \leq p_y \leq \underline{p}_y$ and $\underline{p}_y \leq p_y \leq p_y^0$ to get the global best response.

$$p_y = \begin{cases} \frac{(4V+x\tau+p_x)}{10} & \text{if } 6V - 10\tau - x\tau \leq p_x \leq \frac{16V-11x\tau}{11} \\ 2V - \tau x - p_x & \text{if } \frac{16V-11x\tau}{11} \leq p_x \leq \frac{3}{2}V - \tau x \\ \frac{V}{2} & \text{if } p_x \geq \frac{3}{2}V - \tau x \end{cases}.$$

and the locally optimal location is

$$y = 0.$$

$$\pi_y = \begin{cases} \frac{(4V+x\tau+p_x)^2}{40\tau} & \text{if } 6V - 10\tau - x\tau \leq p_x \leq \frac{16V-11x\tau}{11} \\ \frac{3(x\tau-V+p_x)(2V-x\tau-p_x)}{\tau} & \text{if } \frac{16V-11x\tau}{11} \leq p_x \leq \frac{3}{2}V - \tau x \\ \frac{3V^2}{4\tau} & \text{if } p_x \geq \frac{3}{2}V - \tau x \end{cases}.$$

2. $\tau \leq V \leq \frac{11}{5}\tau$

Putting different parts of the local best responses together gives

$$p_y = \begin{cases} \text{Not defined} & p_x \leq 3V - 7\tau - 5\tau x \\ \frac{3\tau+\tau x+V+p_x}{4} & \text{if } 3V - 7\tau - 5\tau x < p_x \leq 3V - 7\tau - x\tau \\ V - \tau & \text{if } 3V - 7\tau - x\tau \leq p_x \leq 6V - 10\tau - x\tau \\ \frac{(4V+x\tau+p_x)}{10} & \text{if } 6V - 10\tau - x\tau \leq p_x \leq \frac{16V-11x\tau}{11} \\ 2V - \tau x - p_x & \text{if } \frac{16V-11x\tau}{11} \leq p_x \leq \frac{3}{2}V - \tau x \\ \frac{V}{2} & \text{if } p_x \geq \frac{3}{2}V - \tau x \end{cases}.$$

and the locally optimal location is

$$y = \begin{cases} \text{Not defined} & p_x \leq 3V - 7\tau - 5\tau x \\ \frac{-7\tau+3V-x\tau-p_x}{4\tau} & \text{if } 3V - 7\tau - 5\tau x < p_x \leq 3V - 7\tau - x\tau \\ 0 & \text{if } p_x \geq 3V - 7\tau - x\tau \end{cases}.$$

$$\pi_y = \begin{cases} \text{Not defined} & p_x \leq 3V - 7\tau - 5\tau x \\ \frac{(V+3\tau+x\tau+p_x)^2}{16\tau} & \text{if } 3V - 7\tau - 5\tau x < p_x \leq 3V - 7\tau - x\tau \\ \frac{(5\tau-V+x\tau+p_x)(V-\tau)}{2\tau} & \text{if } 3V - 7\tau - x\tau \leq p_x \leq 6V - 10\tau - x\tau \\ \frac{(4V+x\tau+p_x)^2}{40\tau} & \text{if } 6V - 10\tau - x\tau \leq p_x \leq \frac{16V-11x\tau}{11} \\ \frac{3(x\tau-V+p_x)(2V-x\tau-p_x)}{\tau} & \text{if } \frac{16V-11x\tau}{11} \leq p_x \leq \frac{3}{2}V - \tau x \\ \frac{3V^2}{4\tau} & \text{if } p_x \geq \frac{3}{2}V - \tau x \end{cases}.$$

3. $\frac{11}{5}\tau \leq V \leq \frac{77}{17}\tau$ and $x \leq \frac{V-3\tau}{2\tau}$.

Putting different parts of local best responses together, we get

$$p_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq V - \tau(1+x) \\ V - \tau(1+x) & \text{if } V - \tau(1+x) < p_x \leq 3V - 7\tau - 5x\tau \\ \frac{3\tau + \tau x + V + p_x}{4} & \text{if } 3V - 7\tau - 5x\tau \leq p_x \leq 3V - 7\tau - x\tau \\ V - \tau & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

and the locally optimal location is

$$y = \begin{cases} \text{Not defined} & p_x \leq 3V - 7\tau - 5x\tau \\ \frac{-7\tau + 3V - x\tau - p_x}{4\tau} & \text{if } 3V - 7\tau - 5x\tau \leq p_x \leq 3V - 7\tau - x\tau \\ 0 & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

$$\pi_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq V - \tau(1+x) \\ 3(V - \tau(1+x)) & \text{if } V - \tau(1+x) < p_x \leq 3V - 7\tau - 5x\tau \\ \frac{(V + 3\tau + x\tau + p_x)^2}{16\tau} & \text{if} 3V - 7\tau - 5x\tau \leq p_x \leq 3V - 7\tau - x\tau \\ \frac{(5\tau - V + x\tau + p_x)(V-\tau)}{2\tau} & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

4. $\frac{11}{5}\tau \leq V \leq \frac{77}{17}\tau$ and $x \geq \frac{V-3\tau}{2\tau}$.

Putting different parts of local best responses together, we get

$$p_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq 3V - 7\tau - 5\tau x \\ \frac{3\tau + \tau x + V + p_x}{4} & \text{if } 3V - 7\tau - 5\tau x \leq p_x \leq 3V - 7\tau - x\tau \\ V - \tau & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

and the locally optimal location is

$$y = \begin{cases} \text{Not defined} & \text{if } p_x \leq 3V - 7\tau - 5\tau x \\ \frac{-7\tau + 3V - x\tau - p_x}{4\tau} & \text{if } 3V - 7\tau - 5\tau x \leq p_x \leq 3V - 7\tau - x\tau \\ 0 & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

$$\pi_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq 3V - 7\tau - 5\tau x \\ \frac{(V + 3\tau + x\tau + p_x)^2}{16\tau} & \text{if } 3V - 7\tau - 5\tau x \leq p_x \leq 3V - 7\tau - x\tau \\ \frac{(5\tau - V + x\tau + p_x)(V-\tau)}{2\tau} & \text{if } 3V - 7\tau - x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

5. $V \geq \frac{77}{17}\tau$ and $x \leq \frac{V-3\tau}{2\tau}$.

Putting different parts of local best responses together, we get

$$p_y = \begin{cases} \text{Not defined} & \text{if } p_x \leq V - \tau(1+x) \\ V - \tau(1+x) & \text{if } V - \tau(1+x) < p_x \leq 3V - 7\tau - 5x\tau \\ \frac{3\tau + \tau x + V + p_x}{4} & \text{if } 3V - 7\tau - 5x\tau \leq p_x \leq V + \tau(1-x) \end{cases}.$$

and the locally optimal location is

$$y=\begin{cases}\text{Not defined} & p_x\leq 3V-7\tau-5x\tau\\ \frac{-7\tau+3V-x\tau-p_x}{4\tau} & \text{if } 3V-7\tau-5x\tau\leq p_x\leq V+\tau(1-x)\end{cases}.$$

$$\pi_y=\begin{cases}\text{Not defined} & \text{if } p_x\leq V-\tau(1+x)\\ 3(V-\tau(1+x)) & \text{if } V-\tau(1+x)<p_x\leq 3V-7\tau-5x\tau\\ \frac{(V+3\tau+x\tau+p_x)^2}{16\tau} & \text{if} 3V-7\tau-5x\tau\leq\ p_x\leq V+\tau(1-x)\end{cases}.$$

6. $V\geq\frac{77}{17}\tau$ and $x\geq\frac{V-3\tau}{2\tau}$.

Putting different parts of local best responses together, we get

$$p_y=\begin{cases}\text{Not defined} & \text{if } p_x\leq 3V-7\tau-5\tau x\\ \frac{3\tau+\tau x+V+p_x}{4} & \text{if } 3V-7\tau-5\tau x\leq p_x\leq V+\tau(1-x)\end{cases}.$$

and the locally optimal location is

$$y=\begin{cases}\text{Not defined} & \text{if } p_x\leq 3V-7\tau-5\tau x\\ \frac{-7\tau+3V-x\tau-p_x}{4\tau} & \text{if } 3V-7\tau-5\tau x\leq p_x\leq V+\tau(1-x)\end{cases}.$$

$$\pi_y=\begin{cases}\text{Not defined} & \text{if } p_x\leq 3V-7\tau-5\tau x\\ \frac{(V+3\tau+x\tau+p_x)^2}{16\tau} & \text{if } 3V-7\tau-5\tau x\leq p_x\leq V+\tau(1-x)\end{cases}.$$

4.2 The leader's optimal location and pricing

We discuss the first mover's optimisation problem in two cases: $x=0$ and $x>0$, taking into consideration of firm y's best responses characterised in the previous section. Comparing these two cases gives us the global optimum for firm x.

4.2.1 If $x=0$

Firm y's best response is either $y>x$ with pricing behaviour given in Lemma 1 or $y=x=0$ with pricing behaviour given in Lemma 2.

Define the price level which gives $\bar{t}_x=1$ to be $\widetilde{p}_x$.

$$\frac{V-p_x}{\tau}=1\Longrightarrow\widetilde{p}_x=V-\tau.$$

Let $\underline{p}_x$ solves $\bar{t}_x=\underline{t}_y$. By definition, $\widetilde{p}_x<\underline{p}_x$.

For $p_x\leq\widetilde{p}_x$,

$$D_x=2+t_{xy}=2+\frac{p_y-p_x+\tau y}{2\tau}.$$

For $\widetilde{p}_x \leq p_x \leq \underline{p}_x$,

$$D_x = 2\frac{V - p_x}{\tau} + t_{xy} = \frac{4V + y\tau - 5p_x + p_y}{2\tau},$$

For $p_x \geq \underline{p}_x$,

$$D_x = 3\frac{V - p_x}{\tau}.$$

In this section, we take firm x's location fix at $x = 0$ and analyse the optimal p_x according to firm y's best response. With $x = 0$, firm y can choose $y > 0$ or $y = 0$. For every given p_x, firm y can always choose to locate at $y = x = 0$ and get at least the same profit level as firm x by undercutting p_x marginally. Therefore, to make $(x = 0, y > 0)$ an equilibrium, firm x needs to charge a price such that $\pi_y (y = x = 0) \leq \pi_y (x = 0, y > 0)$. Firm x's optimal pricing is summarised in the following lemma with the proof collected in the appendix.

Lemma 3 *For $x = 0$, firm x's local optimal pricing and resulting equilibrium profit is*

$$p_x^* = \begin{cases} \frac{3-\sqrt{3}}{6}V & V \leq \frac{6}{9+\sqrt{3}}\tau \\ \frac{V+\tau-\sqrt{2(8V^2-9V\tau+3\tau^2)}}{5} & \frac{6}{9+\sqrt{3}}\tau \approx 0.56\tau \leq V \leq \frac{63-\sqrt{297}}{68}\tau \\ \frac{23V-\tau-\sqrt{48(2V-\tau)(5V+\tau)}}{49} & \frac{63-\sqrt{297}}{68}\tau \approx 0.673\tau \leq V \leq \left(6-\sqrt{24}\right)\tau \\ 23\tau - V - \sqrt{48\tau\left(11\tau - V\right)} & \left(6-\sqrt{24}\right)\tau \approx 1.1\tau \leq V \leq \frac{11-\sqrt{57}}{2}\tau \approx 1.73\tau \end{cases};$$

$$\pi_x = \begin{cases} \frac{V^2}{2\tau} \\ \frac{3\left(\sqrt{2(8V^2-9V\tau+3\tau^2)}+4V-\tau\right)\left(V-\sqrt{2(8V^2-9V\tau+3\tau^2)}+\tau\right)}{25\tau} \\ \frac{\left(9\sqrt{48(2V-\tau)(5V+\tau)}+136V+156\tau\right)\left(23V-\sqrt{48(2V-\tau)(5V+\tau)}-\tau\right)}{9604\tau} \\ \frac{\left(V-23\tau+4\sqrt{\tau(33\tau-3V)}\right)\left(3\tau-\sqrt{\tau(33\tau-3V)}\right)}{\tau} \end{cases}.$$

Proof. See the appendix. ■

4.2.2 If $x > 0$

With $x \geq y \geq 0$ and $r_x = r_y$, we only need to consider the case that D_x comes from r_x. Define $\widetilde{p}_x$ to be the price level such that $\bar{t}_x = 1$. By definition

$$V - \tau\left(1 - x\right) - \widetilde{p}_x = 0 \Rightarrow \widetilde{p}_x = V - \tau\left(1 - x\right).$$

The optimal $p_x^* = \frac{V}{2}$ if x can act as a local monoplist. Otherwise, $p_x = \bar{p}_x$. The following lemma presents x's optimal pricing and location for the caes $x \geq y \geq 0$ and $r_x = r_y$.

Lemma 4 *For $x > 0$, firm x's optimal pricing and resulting equilibrium profit is*

$$(x, p_x) = \begin{cases} \left(x \in \left[\frac{V}{\tau}, \frac{2\tau - V}{2\tau}\right], \frac{V}{2}\right) & V \leq \frac{2}{3}\tau \\ \left(\frac{2\tau - V}{2\tau}, \frac{V}{2}\right) & \frac{2}{3}\tau \leq V \leq \frac{11}{16}\tau \\ \left(\frac{5V + 11\tau}{22\tau}, \frac{27V - 11\tau}{22}\right) & \frac{11}{16}\tau \leq V \leq \frac{319}{343}\tau \\ \left(\frac{47\tau - 23V}{36\tau}, \frac{13V + 11\tau}{36}\right) & \frac{319}{343}\tau \leq V \leq \frac{47}{23}\tau \end{cases}.$$

$$\pi_x = \begin{cases} \frac{V^2}{2\tau} & V \leq \frac{11}{16}\tau \\ \frac{(27V - 11\tau)(11\tau - 5V)}{242\tau} & \frac{11}{16}\tau \leq V \leq \frac{319}{343}\tau \\ \frac{(13V + 11\tau)^2}{1440\tau} & \frac{319}{343}\tau \leq V \leq \frac{47}{23}\tau \end{cases}.$$

Proof. See the appendix. ■

4.2.3 Optimal x and p_x

We now compare results in Lemmas 3 and 4 to get firm x's optimal location and pricing. Our results show that firm x always prefers to locate off the centre. The intuition is that if firm x locates at the centre, given that firm y can choose to locate at the centre and undercut p_x marginally, the profit firm x gets, in most cases, is equal to the profit of the second mover locating off the centre. However, if firm x chooses to locate off the centre, since firm y would prefer to locate at the centre, the profit level firm x gets is the profit level for the first mover locating off the centre.

Proposition 2 *For $V > \frac{6}{9+\sqrt{3}}\tau \approx 0.56\tau$, firm x always prefers to locate off the centre with the optimal (x, p_x) given in Lemma 4. For $V \leq \frac{6}{9+\sqrt{3}}\tau$, firm x is indifferent between choosing $x = 0$ and $x > 0$ with $\pi_x = \frac{V^2}{2\tau}$ in both cases.*

Proof. See the appendix. ■

5 Three Firm Analysis

For three firm oligopoly, we propose an equilibrium configuration and verify the parameter ranges to support it as an equilibrium. The equilibrium configuration we focus on is $x > 0$, $y > 0$, $z \geq 0$, $r_x \neq r_y \neq r_z$, and the three firms do not behave as local monopolists. With three firms, there is even less incentives for x and y to locate at the centre.

5.1 Firm z's decision

We solve the game backwards starting with firm z's decision. First, it is never optimal to have $\bar{t}_z > 1$. If this is the case, for any given price, z would find it optimal to move towards the centre, since the demand coming from r_z remains the same while the demand coming from the other two rays increase. Similarly, it is never optimal to have $\bar{t}_z < 1$. For any given prices, if $\bar{t}_z < 1$, z can always prefer to move outwards. The demand would remain the same, while the firm would face less competition from the other two firms. Therefore, apart from the case where firms are local monopolists, in equilibrium, (p_z, z) are chosen such that $\bar{t}_z = 1$. For $z \geq 0$, this is defined as

$$V - \tau(\bar{t}_z - z) - p_z = 0.$$

Imposing the condition $t_z = 1$ gives

$$z = \frac{p_z - V + \tau}{\tau}.$$

Let's focus on the equilibrium where (1) D_z comes from r_x and r_y as well (2) firms are not local monopolists. In this case, $D_z = t_{xz} + t_{yz} + 1$. The marginal consumers are defined:

$$V - \tau(x - t_{xz}) - p_x = V - \tau(t_{xz} + z) - p_z.$$

This gives

$$t_{xz} = \frac{\tau(x - z) + p_x - p_z}{2\tau}.$$

Similarly,

$$t_{yz} = \frac{\tau(y - z) + p_y - p_z}{2\tau}.$$

With the constraint, $\bar{t}_z = 1$, the demand is

$$\begin{aligned} D_z &= \frac{\tau(x - z) + p_x - p_z}{2\tau} + \frac{\tau(y - z) + p_y - p_z}{2\tau} + 1 \\ &= \frac{\tau x + V - \tau + p_x - 2p_z}{2\tau} + \frac{\tau y + V - \tau + p_y - 2p_z}{2\tau} + 1. \end{aligned}$$

Firm z solves the problem

$$\max_{p_z} \left(\frac{\tau x + V - \tau + p_x - 2p_z}{2\tau} + \frac{\tau y + V - \tau + p_y - 2p_z}{2\tau} + 1 \right) p_z.$$

¿From FOC,

$$\left(\frac{-2}{2\tau} + \frac{-2}{2\tau} \right) p_z + \frac{\tau x + V - \tau + p_x - 2p_z}{2\tau} + \frac{\tau y + V - \tau + p_y - 2p_z}{2\tau} + 1 = 0,$$

we obtain,

$$p_z(p_x, x, p_y, y) = \frac{\tau x + 2V + p_x + \tau y + p_y}{8}, \tag{3}$$

and

$$z(p_x, x, p_y, y) = \frac{\tau x + \tau y + p_x + p_y - 6V + 8\tau}{8\tau}. \tag{4}$$

These are firm z's best-response correspondences.[2]

5.2 Firm y's decision

We analyse firm y's decision given (p_x, x) and anticipating reactions (3) and (4). First of all, the same argument in th previous section applies and in equilibrium, we have $\bar{t}_y = 1$, or

$$y = 1 - \frac{V - p_y}{\tau}.$$

The demand for firm y is $1 - t_{yz}$ and profit is,

$$\pi_y = (1 - t_{yz})\, p_y = \left(1 - \frac{\tau\,(y - z) + p_y - p_z}{2\tau}\right) p_y$$

Substituting best-responses (3) and (4) and $\bar{t}_y = 1$,

$$\pi_y = \left(\frac{V + \tau + x\tau + p_x - 6p_y + 8\tau}{8\tau}\right) p_y.$$

The optimal price and locations are,

$$p_y(p_x, x) = \frac{V + 9\tau + \tau x + p_x}{12}, \quad y(p_x, x) = \frac{-11V + 21\tau + \tau x + p_x}{12\tau}. \tag{5}$$

5.3 Firm x's decision

Firm x's demand is $1 - t_{xz}$, and profit is,

$$\pi_x = (1 - t_{xz})p_x = \left(1 - \frac{\tau\,(x - z) + p_x - p_z}{2\tau}\right) p_x.$$

Using the best-responses (3), (4) and (5) and restriction $\bar{t}_x = 1$, we have,

$$\pi_x = \frac{35\tau - 17p_x}{24\tau} p_x.$$

[2]The optimal choices for given (p_x, x, p_y, y) are actually price given by (3) and any $0 \leq z \leq \frac{\tau x + \tau y + p_x + p_y - 6V + 8\tau}{8\tau}$.

The optimal price and locations are,

$$p_x^* = \frac{35}{34}\tau, \quad x^* = \frac{69}{34} - \frac{V}{\tau}.$$

¿From (3), (4) and (5), we have the optimal prices and location of the two other firms,

$$p_y^* = \frac{205}{204}\tau, \quad y^* = \frac{409}{204} - \frac{V}{\tau}, \quad p_z^* = \frac{619}{816}\tau, \quad z^* = \frac{1435}{816} - \frac{V}{\tau}.$$

The equilibrium marginal consumers are,

$$t_{xz}^* = \frac{13}{48}, \quad t_{yz}^* = \frac{67}{272}.$$

The equilibrium profits are,

$$\pi_x^* = \frac{1225}{1632}\tau < \pi_y^* = \frac{42025}{55488}\tau < \pi_z^* = \frac{383161}{332928}\tau.$$

5.4 Verification of the support for the equilibrium

For the above configuration to be supported as an equilibrium, the following conditions need to be satisfied.

1. Firms are not local monopolists.
2. With $r_z \neq r_y \neq r_x$, restrict the parameter ranges to the ones where D_z comes from r_x and r_y.
3. Make sure that z does not want to choose to locate on r_x and r_y.
4. Make sure that all natural restrictions are satisfied. That is, prices are non-negative and locations are within 0 and 1 (weekly, although in the porposed equilibrium, all locations should be strictly within 0 and 1). And it should be satisfied that $z < x = y$.

If these conditions are satisfied, the proposed equilibrium should indeed be an equilibirum for this three firm sequentical game.

(1) Exclude the parameter ranges where firms behave as local monopolists.

To do this, for $z > 0$, and $r_z \neq r_x \neq r_y$, we need to make sure that $\underline{t}_z \geq \underline{t}_x$ and $\underline{t}_z \geq \underline{t}_y$ at the equilibirum prices and locations. By definition

$$V - \tau(\underline{t}_z + z) - p_z = 0 \Rightarrow \underline{t}_z = \frac{V - p_z}{\tau} - z,$$

$$V - \tau (x - \underline{t}_x) - p_x = 0 \Rightarrow \underline{t}_x = x - \frac{V - p_x}{\tau},$$

and

$$\underline{t}_y = y - \frac{V - p_y}{\tau}.$$

The conditions $\underline{t}_z \geq \underline{t}_x$ and $\underline{t}_z \geq \underline{t}_y$ are satisfied if

$$\frac{V - p_z}{\tau} - z \geq x - \frac{V - p_x}{\tau} \text{ and } \frac{V - p_z}{\tau} - z \geq y - \frac{V - p_y}{\tau}.$$

Substituting in the prices and locations from our proposed equilibrium gives

$$V \geq \frac{2275}{1632}\tau \approx 1.39\tau.$$

(2) With $r_z \neq r_y \neq r_x$, restrict the parameter ranges to the ones where D_z comes from r_x and r_y.

Here, the comparison should be made with the case $z > 0$, $r_z \neq r_x \neq r_y$, and D_z only comes from r_z. The cases where $r_z = r_x$ or $r_z = r_y$ will be dealt with separately. We argue that due to the sequence of the moves, if D_x is restricted to cover only consumers on r_z, it should be the case that firms would choose $x > y \geq 0$ with D_y coming from all three rays.

When we focus on the cases where firms are not local monopolists, the optimal price and location choice should satisfy the conditions that $\bar{t} = 1$ for all firms. We analyse the game backwards and start with firm z's best response. As analysed previously, $\bar{t}_z = 1$ gives $z = \frac{p_z - V + \tau}{\tau}$. The marginal consumer t_{yz} is determined by

$$V - \tau (z - t_{yz}) - p_z = V - \tau (y + t_{yz}) - p_y.$$

This gives$t_{yz} = \frac{\tau(z-y)+p_z-p_y}{2\tau}$. The demand for firm z is therefore

$$D_z = 1 - t_{yz} = \frac{\tau + V + \tau y - 2p_z + p_y}{2\tau}.$$

The optimisation problem is therefore

$$\max_{p_z} \left(\frac{\tau + V + \tau y - 2p_z + p_y}{2\tau} \right) p_z.$$

The FOC gives

$$p_z = \frac{\tau + V + \tau y + p_y}{4} \text{ and } z = \frac{5\tau - 3V + \tau y + p_y}{4\tau}.$$

Now we turn to firm y's decision. Firm y is in competition with both firms. It takes (p_x, x) as given and it takes into consideration firm z's best response. We focus on the case where D_y comes from all three rays. The condition $\bar{t}_y = 1$ gives $y = \frac{p_y - V + \tau}{\tau}$. The marginal consumer t_{xy}, located on r_x, is $\frac{\tau(x-y)+p_x-p_y}{2\tau}$. The demand for firm y is therefore

$$D_y = 1 + \frac{\tau(x-y) + p_x - p_y}{2\tau} + \frac{\tau(z-y) + p_z - p_y}{2\tau}.$$

Substituting in the condition $\bar{t}_y = 1$ and z's best response gives

$$D_y = 1 + \frac{\tau x + V + p_x - 3p_y}{2\tau}.$$

The optimisation problem for firm y is

$$\max_{p_y} \left(1 + \frac{\tau x + V + p_x - 3p_y}{2\tau}\right) p_y.$$

The FOC and the condition $\bar{t}_y = 1$ give

$$p_y = \frac{2\tau + \tau x + V + p_x}{6} \text{ and } y = \frac{8\tau + \tau x + p_x - 5V}{6\tau}.$$

Finally, we analyse the choice of (p_x, x) taking into consideration both firms y and z's best responses. The condition $\bar{t}_x = 1$ implies $x = \frac{p_x - V + \tau}{\tau}$. Substituting in y's best responses gives

$$D_x = 1 - \frac{4p_x - 3\tau}{6\tau}.$$

The optimisation problem is

$$\max_{p_x} \left(1 - \frac{4p_x - 3\tau}{6\tau}\right) p_x.$$

The FOC and the condition $\bar{t}_x = 1$ give

$$p_x = \frac{9}{8}\tau \approx 1.125\tau \text{ and } x = \frac{17\tau - 8V}{8\tau}.$$

¿From the best responses and $\bar{t} = 1$, the prices and locations for the other two firms are

$$p_y = \frac{7\tau}{8} \approx 0.875\tau,\ y = \frac{15\tau - 8V}{8\tau},\ p_z = \frac{15\tau}{16} \approx 0.937\tau, \text{ and } z = \frac{31\tau - 16V}{16\tau}.$$

The marginal consumers are

$$t_{xy} = \frac{1}{4} \text{ and } t_{yz} = \frac{1}{16}.$$

The resulting profits for the firms are

$$\pi_x = (1 - t_{xy})\, p_x = \frac{27}{32}\tau \approx 0.84\tau,$$

$$\pi_y = (1 + t_{xy} + t_{yz})\, p_y = \frac{147\tau}{128} \approx 1.15\tau.$$

$$\pi_z = (1 - \ t_{yz})\, p_z = \frac{225}{256}\tau \approx 0.88\tau.$$

Firm z gets higher profit when its demand comes from three rays. Furthermore, $\pi_z < \pi_y$. This can never be an equilibirum.

(3) Make sure that z does not want to choose to locate on r_x and r_y.

To achieve this, we impose the restriction that in the above proposed equilibrium, $z > 0$. Note that D_z comes from all three rays in our proposed equilibrium. If in equilibrium, $z > 0$ and $r_z \neq r_x \neq r_y$, z does not have incentive to move into rays x and y. The condition we require is

$$\frac{1435}{816} - \frac{V}{\tau} > 0 \text{ or } V < \frac{1435}{816}\tau \approx 1.76\tau.$$

(4) All the equilibrium prices are positve, $p_i > 0$, and $\{x, y, z\} \in (0, 1)$.

The conditions are prices are satisfied. The restrictions on the positions give

$$0 < \frac{69}{34} - \frac{V}{\tau} < 1,$$

$$0 < \frac{409}{204} - \frac{V}{\tau} < 1,$$

and

$$0 < \frac{1435}{816} - \frac{V}{\tau} < 1.$$

To satisfiy the restions simultaneous, we have

$$\frac{35}{34}\tau \approx 1.03\tau < V < 1.76\tau.$$

Together with the restriction that firms are not local monopolists, we have the parameter range for our proposed equilibrium as

$$\frac{2275}{1632}\tau \approx 1.39\tau \leq V < 1.76\tau.$$